Material Dialogues for First-Order Logic in Constructive Type Theory

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Abstract

Lorenzen's material dialogues are turn taking games which model debates about the satisfaction of formulas in an underlying model. As a novel variant, when played over first-order structures, they give rise to a notion of first-order satisfaction. We study the arising notion of validity for classical and intuitionistic first-order logic in the constructive setting of the calculus of inductive constructions. We prove that this material dialogue semantics for classical first-order logic admits constructive soundness and completeness proofs, setting it apart from standard model theoretic semantics of first-order logic. Furthermore, we demonstrate that completeness proofs with regards to intuitionistic material dialogues fail in the constructive setting. As an alternative, we propose material Kripke dialogues, played over first-order Kripke structures, which admit constructive soundness and completeness proofs when restricted to the $\forall, \rightarrow, \bot$-fragment of intuitionistic first-order logic. The results concerning classical material dialogues have been mechanized using the Coq interactive theorem prover.

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Supplementary Material Mechanization: `https://github.com/dowehr/material-dialogues-coq`

1 Introduction

Logical dialogues were first introduced by Paul Lorenzen [15, 16], a philosopher and constructive mathematician active throughout the latter half of the twentieth century. They constitute a continuation of his search for a constructively acceptable interpretation of logic, beginning with his work on operative mathematics [14]. Logical dialogues are turn-taking games which model a debate in which the proponent defends the validity of a formula against the criticisms of an opponent. As such, the games' moves are modeled after speech acts: asserting formulas and questioning assertions made by the other player. In this, they differ from the nowadays more wide-spread logical games in the style of Hintikka [10], in which a formula is reduced to its atoms by both players, the turn order being determined by the syntactical structure of the formula. Although logical dialogues were initially put forward as a semantics for intuitionistic logic, they can capture classical logic as well.

Any dialogue begins with the proponent asserting a formula to be discussed. The players then take turns, starting with the opponent. The player at turn can choose between two possible moves: Either they can attack an assertion made by the opposing player or they can defend against such an attack targeting one of their assertions, usually by asserting another formula. Some attacks force the attacking player to assert a formula in order to carry out the attack, in which case the assertion in question is called an admission. As an example, attacking the asserted formula $\varphi \rightarrow \psi$ requires the attacker to admit φ while attacking. To defend against the attack on $\varphi \rightarrow \psi$, the attacked player must assert ψ. As usual for games,

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finite plays are won by the player who made the last move. Infinite plays are always won by the opponent. To prevent the opponent from winning by "stalling" indefinitely, for example through repeated attacks on the proponent's initial assertion, additional restrictions are imposed on the opponent's legal moves. The most common such restriction is to forbid the opponent from attacking any of the proponent's admissions twice during a play.

While the meaning of logical connectives can be captured by attacks and defenses allowing the players to break down assertions into subformulas, this approach does not extend to atomic formulas. There are two different approaches to incorporating atomic formulas in logical dialogues. Material dialogues, the variant originally proposed by Lorenzen [15, 16], permit attacks on atomic formulas. To defend against such an attack, the attacked player is required to "demonstrate" the validity of the atomic formula, which in Lorenzen's original formulation meant deriving a word according to a grammar, a remainder of his operative semantics of mathematics [14]. Formal dialogues, which were put forward in the dissertation of Lorenzen's student Kuno Lorenz [13], treat atomic formulas without appealing to their "underlying meaning", i.e. by purely syntactic means. Atomic formulas cannot be attacked by either player and an additional restriction is imposed on the proponent: They may only assert those atomic formulas which the opponent has asserted previously. Historically, the study of logical dialogues after Lorenzen's and Lorenz' initial work has been focused on formal dialogues due to their greater simplicity [12]. Indeed, Sørensen and Urzyczyn [18] have demonstrated that the winning strategies of formal dialogues for propositional logic are structurally similar to sequent calculus derivations, a result which we have extended to first-order logic in [5].

If one fixes the method of justifying the claim of an atomic formula in a material dialogue to be its satisfaction in a previously agreed upon first-order structure, this gives rise to model-theoretic notions of satisfaction and, by quantifying over all models, validity. In this paper, we study the arising semantics of first-order logic in the constructive setting of the calculus of inductive constructions [1, 17]. This extends our previous investigation [5, 6] into the constructivity of completeness theorems for various semantics of first-order logic, including among them formal dialogues for intuitionistic first-order logic.

1.1 Outline and Contributions of the Paper

In this section, we summarize the paper's results. Section 2 only covers some basic definitions and results used throughout the paper. Note that our use of "soundness" and "completeness" is somewhat unusual. When speaking of the soundness of a semantics, the intended meaning is the soundness of some suitable first-order deduction system with regards to said semantics (and similarly for "completeness"). While non-standard, this terminology allows us to be more concise about this paper's results.

Classical material dialogues In Section 3, we lay out material dialogues for classical first-order logic with all connectives. In Section 3.1, we prove their soundness with regards to a cut-free, classical sequent calculus. Notably, classical material dialogues are sound on any first-order structure, whereas classical Tarski semantics require the underlying structure to satisfy all instances of the law of the excluded middle (LEM), a property not necessarily held by all structures in a constructive setting. One could thus say that the "classicality" of classical material dialogues is within their rules of engagement, not the underlying structures. In Section 3.2 we prove that classical material validity entails exploding classical Tarski validity, a constructively stricter notion than standard classical Tarski validity. We then use the constructive completeness of exploding classical Tarski validity on the $\forall, \rightarrow, \bot$-fragment [8] to deduce the same for classical material dialogues. We constructively extend the completeness result for classical material dialogues to the full syntax of first-order logic via the DeMorgan translation, a result that requires the full LEM for the standard Tarski semantics. The results of Section 3 have been mechanized in the interactive theorem prover Coq.

Intuitionistic material dialogues In Section 4 we analyze material dialogues for first-order logic with the usual rules for dialogues of intuitionistic logics. We prove that standard Tarski validity on the fragment $\mathtt{F}^{\mathtt{D}}$ given below entails intuitionistic material validity.

$$a, b : \mathtt{A} ::= \bot \mid P\,\bar{t} \mid a \wedge b \mid a \vee b \mid \exists x.a \qquad P : \Sigma, \bar{t} : \mathtt{T}, x : \mathbf{V}$$
$$\varphi, \psi : \mathtt{F}^{\mathtt{D}} ::= a \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid a \rightarrow \psi \mid \forall x.\varphi \mid \exists x.\varphi \qquad x : \mathbf{V}$$

This means that proving completeness with regards to intuitionistic material dialogues is tantamount to disproving non-constructive principles such as the LEM on that fragment. As such principles are consistent with most constructive meta-theories, including the CIC, completeness cannot be established without additional axioms. The result culminates in the fact that under the LEM, intuitionistic and classical material dialogues completely coincide. Intuitionistic material dialogues are thus ill-suited as a semantics of intuitionistic first-order logic in common constructive settings.

Material Kripke dialogues In reaction to the results of Section 4, we propose an alternative dialogical semantics in Section 5. As classical material dialogues could be considered "classical dialogues played on the canonical structure of classical semantics", we consider the intuitionistic analogue: intuitionistic dialogues played on Kripke structures. We demonstrate their suitability by deriving many of the same results for them as for the classical material dialogues of Section 3. In Section 5.1 we prove them sound with regards to a cut-free intuitionistic sequent calculus. In Section 5.2, we show that material Kripke validity entails exploding Kripke validity. We use the constructive completeness of exploding Kripke models for the $\forall, \rightarrow, \bot$-fragment [9] to deduce the same for material Kripke dialogues. To derive completeness with regards to all connectives, the additional assumption of the Fan theorem seems to be required [19].

2 Preliminaries

2.1 The Calculus of Inductive Constructions

The results of this paper are all derived within the Calculus of Inductive Constructions (CIC) [1, 17], the type theory underlying the interactive theorem prover Coq. The CIC consists of a predicative hierarchy of type universes $\mathbb{T}_i$ above an impredicative universe $\mathbb{P}$ of propositions. Each type universe contains an empty type, products $A \times B$, sums $A + B$, function types $A \rightarrow B$, dependent products $\Pi a : A.B(a)$ and dependent products $\Sigma a : A.B(a)$. In $\mathbb{P}$, we denote them by their respective Curry-Howard correspondents $\bot, \wedge, \vee, \rightarrow, \forall, \exists$. Allowing unrestricted elimination from $\mathbb{P}$ into the $\mathbb{T}_i$ results in an inconsistency [7]. However, this restriction can be lifted for some types in $\mathbb{P}$, including types of at most one constructor, such as $\bot$ and the equality type $= : \Pi A.A \rightarrow A \rightarrow \mathbb{P}$ with a sole constructor $\forall (a : A).\ a = a$.

We use inductive types for the natural numbers ($n : \mathbb{N} := 0 \mid S\,n$), option types ($\mathcal{O}(A) := \ulcorner a \urcorner \mid \emptyset$) and list types ($l : \mathcal{L}(A) := [] \mid a :: l$). We denote *list membership* by $a \in l$ and the *list appending operation* by $l +\!\!+ l'$.

2.2 First-Order Predicate Logic

Fix a *signature* Σ of n-ary functions f and predicates P. Let $\mathbf{V}$ be the type of countably many variables $x, y, z : \mathbf{V}$, then we define an associated *term and formula language.*

$$\mathtt{T} ::= x \mid f\,\vec{t} \qquad\qquad x : \mathbf{V}, c : \Sigma, \vec{t} : \mathtt{T}^{|f|}$$

$$\varphi : \mathtt{F} ::= \bot \mid P\,\vec{t} \mid \varphi \land \psi \mid \varphi \lor \psi \mid \varphi \to \psi \mid \forall x.\varphi \mid \exists x.\varphi \qquad x : \mathbf{V}, P : \Sigma, \vec{t} : \mathtt{T}^{|P|}$$

Negation is defined as a shorthand $\neg\varphi := \varphi \to \bot$. Note that formally, especially within proofs, we are working with DeBruijn binders [3] instead of the syntax with named binders we present here. However, for the sake of readability, we opt to present all definitions in the main text of this paper in the familiar style of named binders. For more details on DeBruijn binders, including the changes of definitions required by them, the reader is invited to consult Appendix A.

A *structure* $\mathbf{S}$ consists of a type X, a predicate interpretation $P^{\mathbf{S}} : X \to ... \to X \to \mathbb{P}$ for each $P \in \Sigma$, a function interpretation $f^{\mathbf{S}} : X \to ... \to X \to X$ for each $f \in \Sigma$ and an absurdity interpretation $\bot^{\mathbf{S}} : \mathbb{P}$. A *model* $\mathbf{M}$ consists of a structure $\mathbf{S}$ together with an assignment $\rho : \mathbf{V} \to X$. We define the usual term evaluation function t^ρ inside a structure $\mathbf{S}$

$$x^\rho := \rho\, x \qquad (f\,\vec{t})^\rho := f^{\mathbf{S}}\,\vec{t}^\rho$$

and the Tarski satisfaction relation $\rho \vDash \varphi$

$$\rho \vDash P\,\vec{t} :\Leftrightarrow P^{\mathbf{S}}\,\vec{t}^\rho \qquad \rho \vDash \varphi \to \psi :\Leftrightarrow \rho \vDash \varphi \to \rho \vDash \psi$$
$$\rho \vDash \varphi \land \psi :\Leftrightarrow \rho \vDash \varphi \land \rho \vDash \psi \qquad \rho \vDash \varphi \lor \psi :\Leftrightarrow \rho \vDash \varphi \lor \rho \vDash \psi$$
$$\rho \vDash \forall x.\varphi :\Leftrightarrow \forall s : \mathbf{S},\ \rho[x \mapsto s] \vDash \varphi \qquad \rho \vDash \exists x.\varphi :\Leftrightarrow \exists s : \mathbf{S},\ \rho[x \mapsto s] \vDash \varphi$$
$$\rho \vDash \bot :\Leftrightarrow \bot^{\mathbf{S}}$$

We call a structure $\mathbf{S}$ *classical* if for all environments ρ and formulas φ, ψ it can be shown to satisfy the principle of double-negation elimination ($\rho \vDash \neg\neg\varphi \to \varphi$). We call a structure $\mathbf{S}$ *exploding* if for all environments ρ and formulas φ it can be shown to satisfy the principle of Explosion ($\rho \vDash \bot \to \varphi$). We call a structure $\mathbf{S}$ *standard* if $\bot^{\mathbf{S}}$ is contradictory (i.e $\neg\bot^{\mathbf{S}}$). Observe that all standard structures are exploding. We call a formula φ *valid in classical exploding structures* in the context of a finite list of formulas Γ, written $\Gamma \vDash^E \varphi$, if for all classical, exploding structures $\mathbf{S}$ and all environments ρ we have that $\rho \vDash \Gamma$ entails $\rho \vDash \varphi$. *Validity in classical standard structures*, $\Gamma \vDash^S \varphi$, is defined analogously.

A *Kripke frame* consists of a type K and a preorder $\leq : K \to K \to \mathbb{P}$ on that type which is a mere proposition, i.e. $H = H'$ for any $H, H' : k \leq k'$. A *Kripke structure* consists of a Kripke frame $(K, \leq)$, a family $\mathbf{S} : \Pi k : K.\mathcal{S}$ of structures and a family of functions $\iota : \Pi k, k' : K.k \leq k' \to \mathbf{S}_k \to \mathbf{S}_{k'}$ such that $P^k\,\vec{s}$ entails $P^{k'}\,(\iota(k \leq k')(\vec{s}))$ and $\iota(k' \leq k'')(\iota(k \leq k')(s)) = \iota(k \leq k'')(s)$ for all $k, k', k'' : K$ and $s : \mathbf{S}_k$ when writing P^k as a shorthand for $P^{\mathbf{S}_k}$. For $k : K$ and $s : \mathbf{S}_k$ we write $s^{k'}$ for $\iota\,(k \leq k')\,s$ where applicable, extending it to environments $\rho^k : \mathbf{V} \to \mathbf{S}_k$ via $\rho^{k'} := \iota(k \leq k') \circ \rho^k$. A *Kripke model* consists of a Kripke structure, a world k and an environment ρ^k. Given a Kripke model, we define a relation $\rho^k \Vdash \varphi$, denoting that φ is forced at a world k under the k-environment ρ^k as below.

$$\rho^k \Vdash P\,\vec{t} :\Leftrightarrow P^k\,\vec{t}^\rho \qquad \rho^k \Vdash \varphi \to \psi :\Leftrightarrow \forall k \leq k'.\ \rho^{k'} \Vdash \varphi \to \rho^{k'} \Vdash \psi$$
$$\rho^k \Vdash \varphi \land \psi :\Leftrightarrow \rho^k \Vdash \varphi \land \rho^k \Vdash \psi \qquad \rho^k \Vdash \varphi \lor \psi :\Leftrightarrow \rho^k \Vdash \varphi \lor \rho^k \Vdash \psi$$
$$\rho^k \Vdash \exists x.\varphi :\Leftrightarrow \exists s : \mathbf{S}_k,\ \rho^k[x \mapsto s] \Vdash \varphi \qquad \rho^k \Vdash \forall x.\varphi :\Leftrightarrow \forall k \leq k', s : \mathbf{S}_{k'},\ \rho^{k'}[x \mapsto s] \Vdash \varphi$$
$$\rho^k \Vdash \bot :\Leftrightarrow \bot^k$$

We call a Kripke structure exploding or standard if all $\mathbf{S}_k$ are such. We call a formula φ *valid in exploding models* in a context Γ, written $\Gamma \Vdash^E \varphi$, if for all exploding Kripke structures, all worlds $k : K$ and all k-environments ρ we have that $\rho^k \Vdash \Gamma$ entails $\rho^k \Vdash \varphi$. We denote the analogous validity in standard Kripke models with $\Gamma \Vdash^S \varphi$.

Now let $\vdash^C$ and $\vdash^I$ denote suitable proof systems for classical and intuitionistic first-order logic, respectively. The following results stem from [8] and [9], respectively, and have been mechanized as part of [5]:

▶ **Theorem 1.** *Restricting to the* $\forall, \rightarrow, \bot$*-fragment of first-order logic, the following hold:*
1. *For classical exploding structures,* $\Gamma \vDash^E \varphi$ *entails* $\Gamma \vdash^C \varphi$.
2. *For exploding Kripke structures,* $\Gamma \Vdash^E \varphi$ *entails* $\Gamma \vdash^I \varphi$.

3 Classical Material Dialogues

We begin by giving a formal rendering of material dialogues for classical first-order logic. Fix a standard structure $\mathbf{S}$. The attack moves and corresponding defenses are laid out in Figure 1. We take $\mathbf{A}$ to be the type of attacks and write $a \rhd \varphi$ if a is an *attack on* φ. Some attacks force the attacker to *admit* a formula. This is formalized through a function $\mathsf{adm} : \mathbf{A} \rightarrow \mathcal{O}(\mathtt{F})$ where $\mathsf{adm}\, a = \ulcorner\varphi\urcorner$ means that φ must be admitted when attacking with a and $\mathsf{adm}\, a = \emptyset$ means no admission needs to be made. The type $\mathbf{D}$ of defenses, defined below, features three different kinds of defenses: $D_A\, \varphi$ denotes the act admitting of the formula φ, $D_W\, \varphi(x)\, s$ denotes admitting $\varphi(s)$ where $s : \mathbf{S}$. Lastly, $D_M\, \varphi$ means claiming to be able to demonstrate that φ holds. Note that $D_M\, \varphi$ is only ever instantiated with atomic φ.

$$\mathbf{D} ::= D_A\, \varphi \mid D_W\, \varphi(x)\, s \mid D_M\, \varphi \qquad\qquad \varphi : \mathtt{F}, s : \mathbf{S}$$

Each $a : \mathbf{A}$ has an associated set $\mathcal{D}_a$ of *defenses against* a. We define $\mathsf{adm}\, (A_\rightarrow\, \varphi\, \psi) = \ulcorner\varphi\urcorner$ and $\mathsf{adm}\, a = \emptyset$ for all other attacks.

$$\begin{array}{rl}
A_\bot \rhd \bot & \mathcal{D}_{A_\bot} = \{D_M\, \bot\} \\
A_P\, \vec{t} \rhd P\, \vec{t} & \mathcal{D}_{A_P\, \vec{t}} = \{D_M\, P\, \vec{t}\} \\
A_\rightarrow\, \varphi\, \psi \rhd \varphi \rightarrow \psi & \mathcal{D}_{A_\rightarrow\, \varphi\, \psi} = \{D_A\, \psi\} \\
A_\vee\, \varphi\, \psi \rhd \varphi \vee \psi & \mathcal{D}_{A_\vee\, \varphi\, \psi} = \{D_A\, \varphi, D_A\, \psi\} \\
A_L\, \varphi \rhd \varphi \wedge \psi & \mathcal{D}_{A_L\, \varphi} = \{D_A\, \varphi\} \\
A_R\, \psi \rhd \varphi \wedge \psi & \mathcal{D}_{A_R\, \psi} = \{D_A\, \psi\} \\
A_s\, \varphi(x) \rhd \forall x.\varphi(x) & \mathcal{D}_{A_s\, \varphi(x)} = \{D_W\, \varphi(x)\, s\} \\
A_\exists\, \varphi(x) \rhd \exists x.\varphi(x) & \mathcal{D}_{A_\exists\, \varphi(x)} = \{D_W\, \varphi(x)\, s \mid s : \mathbf{S}\}
\end{array}$$

Figure 1 Attacks and defenses for first-order logic

Classical material dialogues are a turn taking game between two players. The *proponent* tries to defend the satisfaction of some formula, whereas the *opponent* tries to challenge the proponent's claims in such a way that they cannot respond. All dialogues we consider are so-called *E-dialogues* which restrict the opponent to only ever react to the proponent's previous move. It can be shown that the notion of satisfaction induced by E-dialogues is equivalent to that of the more intuitive D-dialogues, in which this restriction is lifted [4, 6].

We model the dialogue game as a state transition system. Fixing a standard structure $\mathbf{S}$, we call a triple $(\rho, A, C) : (\mathbf{V} \to \mathbf{S}) \times \mathcal{L}(\mathtt{F}) \times \mathcal{L}(\mathbf{A})$ a *state*. Together $\mathbf{S}, \rho$ form the *ambient model*. The list A contains all of the opponent's *admissions* while C records all attacks that the opponent has leveled against the proponent.

Each round, the proponent gets to make a move (defined in Figure 2), either challenging one of the opponent's admissions ($PA\,a$) or defending against a challenge previously issued by the opponent, either by admitting a formula ($PD\,\varphi$) or by demonstrating that an atomic formula holds in the ambient model ($PM\,\varphi$). We define the effect of a proponent's defense d on the game state as a function d^P as below, where x' denotes a variable which does not occur in $A, C, \varphi(x)$ and which can be computed from them in some deterministic manner.

$$d^P\,(\rho, A, C) = \begin{cases} (\rho[x' \mapsto s], A, C) & \text{if } d = D_W\,\varphi(x)\,s \\ (\rho, A, C) & \text{otherwise} \end{cases}$$

Similarly we define a function which maps defenses to the proponent move that needs to be made to carry out said defense.

$$\text{move}\,(D_A\,\varphi) = PD\,\varphi \qquad \text{move}\,(D_W\,\varphi(x)\,s) = PD\,\varphi(x') \qquad \text{move}\,(D_M\,\varphi) = PM\,\varphi$$

Lastly, we define when a defense is justified by the ambient model. While $D_A\,\varphi$ and $D_W\,\varphi\,s$ are always justified, $D_M\,\varphi$ requires $\rho \vDash \varphi$ to hold. These definitions allow us to give a compact definition of the state transitions a proponent can trigger by making a move.

$$\text{PA}\,\frac{\varphi \in A \quad a \rhd \varphi}{(\rho, A, C) \rightsquigarrow_p (\rho, A, C)\,;PA\,a} \qquad \text{PD}\,\frac{c \in C \quad d \in \mathcal{D}_c \quad \rho \text{ justifies } d}{(\rho, A, C) \rightsquigarrow_p d^P\,(\rho, A, C)\,;\text{move}\,d}$$

Figure 2 The rules for proponent moves

The opponent must react to the proponent's previous move. If the proponent defended by admitting a formula, they must issue a new challenge against that formula (OA). If the proponent attacked one of their admissions, they can either defend against that attack (OD) or counter the attack, attacking the admission made by the proponent while attacking (OC). If the proponent demonstrated the validity of an atomic formula in the ambient model, the opponent cannot respond at all. We define an operation d^O analogously to that for the proponent and use it to define the transition steps the opponent can trigger by making a move. In a slight abuse of notation, we write $c :: A$, where c is an attack, for $\psi :: A$ if $\mathsf{adm}\,c = \ulcorner\psi\urcorner$ and A if $\mathsf{adm}\,c = \emptyset$.

$$d^O\,(\rho, A, C) = \begin{cases} (\rho, \varphi :: A, C) & \text{if } d = D_A\,\varphi \\ (\rho[x' \mapsto s], \varphi(x') :: A, C) & \text{if } (D_W\,\varphi(x)\,s) \\ (\rho, A, C) & \text{if } d = D_M\,\varphi \end{cases}$$

$$\text{OA}\,\frac{c \rhd \varphi}{(\rho, A, C)\,;PD\,\varphi \rightsquigarrow_o (\rho, \psi :: A, c :: C)} \qquad \text{OC}\,\frac{a \rhd \varphi \quad \mathsf{adm}\,a = \ulcorner\psi\urcorner \quad \psi \rhd c}{(\rho, A, C)\,;PA\,a \rightsquigarrow_o (\rho, c :: A, c :: C)}$$

$$\text{OD}\,\frac{d \in \mathcal{D}_a \quad \rho \text{ justifies } d}{(\rho, A, C)\,;PA\,a \rightsquigarrow_o d^O\,(\rho, A, C)}$$

A state can be *won* if the proponent can ensure the play always eventually ends. We define this as an inductive predicate which is very similar to the type-theoretical rendering of the well-foundedness of a relation.

$$\frac{s \rightsquigarrow_p s' \,;\, m \quad \forall s''.\ s' \,;\, m \rightsquigarrow_o s'' \to \mathrm{Win}\, s''}{\mathrm{Win}\, s}$$

We extend the notion to *winning formulas* φ with $\mathrm{Win}\,(\rho, A, C, \varphi)$ meaning that for all attacks $c \rhd \varphi$ we have $\mathrm{Win}\,(\rho, c :: A, c :: C)$. A formula φ is *valid* in a context Γ, written $\Gamma \vDash^D \varphi$, if for all standard structures $\mathbf{S}$ and environments $\rho : \mathbf{V} \to \mathbf{S}$ we have $\mathrm{Win}\,(\rho, \Gamma, [], \varphi)$.

▶ **Example 2.** Depicted below is a strategy establishing $\mathrm{Win}\,(\rho, [P \to Q, P], [A_Q])$ and thus $\mathrm{Win}\,(\rho, [P \to Q, P], [], Q)$. The proponent first attacks $P \to Q$ and then, in both branches, forces the opponent to demonstrate the validity of an atomic formula, copying that demonstration to win the game. This strategy is can be played on any model $\mathbf{S}, \rho$, meaning it suffices to establish $P \to Q, P \vDash^D Q$.

$$\begin{tikzcd}
 & (\rho, [P \to Q, P], [A_Q]) \arrow[d, "\mathrm{PA}"'] & \\
 & (\rho, [P \to Q, P], [A_Q])\,;\, \mathit{PA}\,(A_{\to}\, P\, Q) \arrow[dl, "\mathrm{OC}"'] \arrow[dr, "\mathrm{OD}"] & \\
(\rho, [P \to Q, P], [A_Q, A_P]) \arrow[d, "\mathrm{PA}"'] & & (\rho, [P \to Q, P, Q], [A_Q]) \arrow[d, "\mathrm{PA}"'] \\
(\rho, [P \to Q, P], [A_Q, A_P])\,;\, \mathit{PA}\, A_P \arrow[d, "\mathrm{OD}"', "\text{establishes } \rho \vDash P"] & & (\rho, [P \to Q, P, Q], [A_Q])\,;\, \mathit{PA}\, A_Q \arrow[d, "\mathrm{OD}"', "\text{establishes } \rho \vDash Q"] \\
(\rho, [P \to Q, P], [A_Q, A_P]) \arrow[d, "\mathrm{PD}"', "\rho \vDash P"] & & (\rho, [P \to Q, P, Q], [A_Q]) \arrow[d, "\mathrm{PD}"', "\rho \vDash Q"] \\
(\rho, [P \to Q, P], [A_Q, A_P])\,;\, \mathit{PM}\, P & & (\rho, [P \to Q, P, Q], [A_Q])\,;\, \mathit{PM}\, Q
\end{tikzcd}$$

3.1 Soundness

We prove that classical material dialogues are sound with regards to a cut-free classical sequent calculus (as defined in Appendix B). This is the easiest soundness result to obtain because of the structural similarity of winning strategies for dialogues and cut-free sequent calculus derivations demonstrated in [18, 5]. Recall that all proofs in this paper, including that of Theorem 4, work with DeBruijn syntax, the details of which can be found in Appendix A. Note further that the results from Sections 3.1 and 3.2 were mechanized in Coq.

While the majority of the soundness proof is straight-forward, some difficulty is introduced by the difference in how quantifier instantiations are treated by the sequent calculus and the material dialogues. Compare the L$\forall$ rule from Appendix B with the state transition caused by the proponent attacking the admission $\forall x.\varphi(x) \in \Gamma$ with $A_{t^\rho}\, \varphi(x)$ and the opponent responding with the only legal defense, both given below. In the case of the sequent calculus, the formula is simply instantiated as $\varphi[t/x]$ whereas in the material dialogue, the instantiation is carried out via the environment ρ.

$$\mathrm{L}\forall \frac{\forall x.\varphi \in \Gamma \quad \Gamma, \varphi[t/x] \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (\rho, \Gamma, C) \rightsquigarrow_{po} (\rho[x' \mapsto t^\rho], \varphi(x') :: \Gamma, C)$$

To prove soundness, we need to show that these two methods of instantiation are "essentially the same". For this, we introduce *congruence relations* on different aspects of dialogues. Given environments ρ, ρ' and formulas φ, φ', we define an equivalence relation $\rho, \varphi \equiv \rho', \varphi'$ as below. Intuitively, $\rho, \varphi \equiv \rho', \varphi'$ means that φ and φ' are equal up to term evaluations in the respective environments. Here $\varphi \Box \psi$ is a place-holder for all binary connectives and $\Box x.\varphi$ for both quantifiers.

$$\frac{}{\rho, \bot \equiv \rho', \bot} \qquad \frac{\vec{t}^{\rho} = \vec{t'}^{\rho'}}{\rho, P\,\vec{t} \equiv \rho', P\,\vec{t'}} \qquad \frac{\rho, \varphi \equiv \rho', \varphi' \quad \rho, \psi \equiv_f \rho', \psi'}{\rho, \varphi \Box \psi \equiv \rho', \varphi' \Box \psi'}$$

$$\frac{\forall d.\ \rho[x \mapsto d], \varphi \equiv \rho'[y \mapsto d], \varphi'}{\rho, \Box x.\varphi \equiv \rho', \Box y.\varphi'}$$

We then extend this congruence to attacks and defenses and show that these relations do indeed "act as congruences" (see Appendix C for details). These congruences give rise to Lemma 3 that is crucial for the proof of soundness.

▶ **Lemma 3.** *Let (ρ, A, C) and (ρ', A', C') be dialogue states such that $\rho, A \equiv \rho', A'$ and $\rho, C \equiv \rho', C'$. If $Win(\rho, A, C)$ then $Win(\rho', A', C')$ as well.*

▶ **Theorem 4** (Soundness). *Let Γ, φ be such that $\Gamma \Rightarrow \varphi$. Then $\Gamma \vDash^D \varphi$.*

Proof. For this, it suffices to show that for any standard structure $\mathbf{S}$ we have

$$\begin{aligned} &\Gamma \Rightarrow \Delta \to \forall \rho, A, C.\ (\forall \delta \in \Delta. \exists c \in C.\ c \rhd \delta \wedge (\forall \psi.\ \mathsf{adm}\, c = \ulcorner \psi \urcorner \to \psi \in A)) \\ &\qquad \to \Gamma \subseteq A \to \mathrm{Win}\,(\rho, A, C) \end{aligned}$$

As this means that $\Gamma \Rightarrow \varphi$ entails $\mathrm{Win}\,(\rho, c :: \Gamma, [c])$ for any $c \rhd \varphi$ and thus $\mathrm{Win}\,(\rho, \Gamma, [], \varphi)$.

We prove the claim per induction on $\Gamma \Rightarrow \Delta$ and only spell out an exemplary subset of the cases. For a full proof, the reader may consult the accompanying mechanization.

L$\to$: Then $\varphi \to \psi \in \Gamma$ and we obtain inductive hypotheses for $\Gamma \Rightarrow \varphi, \Delta$ and $\Gamma, \psi \Rightarrow \Delta$. The proponent thus attacks the admission $\varphi \to \psi$. The opponent has two ways of responding to this attack:
- If the opponent defends against the attack by admitting ψ then the proponent can win by playing the strategy obtained from the inductive hypothesis on $\Gamma, \psi \Rightarrow \Delta$.
- If the opponent counters, attacking the admission φ with some challenge $c \rhd \varphi$ then the proponent can win by playing the strategy obtained from the inductive hypothesis on $\Gamma \Rightarrow \varphi, \Delta$.

L$\forall$: Then $\forall \varphi \in \Gamma$ and we have an IH for $\Gamma, \varphi[t] \Rightarrow \Delta$ for some term t. The proponent thus attacks $\forall \varphi$ with $A_{t^\rho}\, \varphi$ against which the opponent must defend with $D_W\, \varphi\, t^\rho$. The state resulting from this is $(t^\rho \cdot \rho, \varphi :: \uparrow A, \uparrow C)$. However, the IH only yields a winning strategy for the state $(\rho, \varphi[t] :: A, C)$. We can now apply Lemma 3 as $t^\rho \cdot \rho, \varphi :: \uparrow A \equiv_f \rho, \varphi[t] :: A$ and $t^\rho \cdot \rho, \uparrow C \equiv_a \rho, C$ to transform the winning strategy provided by the IH into the desired form. ◀

3.2 Completeness

We first prove completeness for the $\forall, \to, \bot$-fragment of first-order logic and then extend this result to the full syntax by using a DeMorgan translation.

We begin by proving that dialogical validity entails classical, exploding Tarski validity.

▶ **Lemma 5.** *For any Γ and φ, $\Gamma \vDash^D \varphi$ entails $\Gamma \vDash^E \varphi$.*

Proof. Fix a classical, exploding structure $\mathbf{S}$. We extend Tarski satisfaction to defenses via:

$$\rho \vDash D_A\,\varphi \Leftrightarrow \rho \vDash \varphi \qquad \rho \vDash D_W\,\varphi\,s \Leftrightarrow s \cdot \rho \vDash \varphi \qquad \rho \vDash D_M\,\varphi \Leftrightarrow \rho \vDash \varphi$$

Furthermore, we define an auxiliary predicate $\Gamma \vDash_\rho \bigvee \Delta$ on contexts Γ, environments ρ and sets of defenses Δ which intuitively states that under the environment ρ, Γ entails the disjunction of all semantic interpretations of Δ.

$$\Gamma \vDash_\rho \bigvee \Delta \Leftrightarrow \rho \vDash \Gamma \to \forall \vec{s} : \mathbf{S}, \alpha.\ (\forall d \in \Delta.\ \rho \vDash d \to (\vec{s} \cdot \rho) \vDash \alpha) \to (\vec{s} \cdot \rho) \vDash \alpha$$

Now we prove two results:
(1) If for all $c \rhd \varphi$ we have $(c :: \Gamma) \vDash_\rho \bigvee(\mathcal{D}_c \cup \Delta)$ then $\Gamma \vDash_\rho \bigvee(\{D_A\,\varphi\} \cup \Delta)$.
(2) Win (ρ, Γ, C) entails $\Gamma \vDash_\rho \bigvee(\bigcup_{c \in C} \mathcal{D}_c)$
Their proofs have been relegated to Appendix D.1. To prove the original claim, assume $\Gamma \vDash^D \varphi$ and pick some classical, exploding model $\mathbf{S}, \rho$ with $\rho \vDash \Gamma$. Per assumption Win $(\rho, \Gamma, [], \varphi)$ meaning $(c :: \Gamma) \vDash_\rho \bigvee \mathcal{D}_c$ for all $c \rhd \varphi$ by (1) and thus $\Gamma \vDash_\rho \bigvee\{D_A \varphi\}$ by (2). By picking φ as α we then obtain $\rho \vDash \varphi$ as desired. ◀

▶ **Corollary 6.** *$\Gamma \vDash^D \varphi$ entails $\Gamma \vdash \varphi$ when restricting to the $\forall, \to, \bot$-fragment.*

Proof. First notice that the proof of Lemma 5 remains correct when the syntax is restricted to the $\forall, \to, \bot$-fragment, therefore $\Gamma \vDash^E \varphi$. Then $\Gamma \vdash \varphi$ follows by Theorem 1. ◀

To extend Corollary 6 to the full fragment of first-order logic, we employ a DeMorgan translation into the $\forall, \to, \bot$-fragment, given below.

$$\bot^D := \bot \qquad (P\,\vec{s})^D := P\,\vec{s} \qquad (\varphi \to \psi)^D := \varphi^D \to \psi^D \qquad (\varphi \wedge \psi)^D := \neg(\varphi^D \to \neg\psi^D)$$

$$(\varphi \vee \psi)^D := \neg\varphi^D \to \psi^D \qquad (\forall\varphi)^D := \forall\varphi^D \qquad (\exists\varphi)^D := \neg(\forall\neg\varphi^D)$$

We then show that winning strategies are preserved under double-negation translation.

▶ **Lemma 7.** *If Win (ρ, A, C, φ) then Win $(\rho, A^D, C, \varphi^D)$.*

Proof. This proof requires two auxiliary results whose proofs are in Appendix D.2:
(1) Win $(\rho, A +\!\!+ \varphi :: A', C)$ and Win $(\rho, A +\!\!+ A', C, \varphi)$ entail Win $(\rho, A +\!\!+ A', C)$ for any φ
(2) If Win (ρ, A, C) and $A \subseteq A'$, $C \subseteq C'$ then Win (ρ, A', C')
The first principle is a dialogical equivalent of cut-admissibility and the second of the weakening rule common to sequent calculi. The proof now proceeds in two steps. We first show that Win (ρ, A^D, C, φ) and from this that Win $(\rho, A^D, C, \varphi^D)$.

1. We prove the generalization Win $(\rho, A +\!\!+ B, C, \varphi) \to$ Win $(\rho, A +\!\!+ B^D, C, \varphi)$ per induction on B. The case of $B = []$ is trivial, thus suppose $B = \psi :: B'$. By (2) we know that Win $(\rho, A +\!\!+ \psi :: \psi^D :: B', C, \varphi)$. It is well known that $\psi^D \Rightarrow \psi$ and by Theorem 4 and (2) thus Win $(\rho, A +\!\!+ \psi^D :: B', C, \psi)$. We may thus apply (1) to cut ψ and obtain Win $(\rho, A +\!\!+ \psi^D :: B', C, \varphi)$. We then continue per inductive hypothesis with the choice $A = A +\!\!+ [\psi^D]$.
2. It is well known that $\varphi \Rightarrow \varphi^D$ thus Win $(\rho, \varphi :: A^D, C, \varphi^D)$ and by Theorem 4 and (2). We may now apply (1) with Win (ρ, A^D, C, φ) to cut φ and obtain Win $(\rho, A^D, C, \varphi^D)$. ◀

▶ **Theorem 8** (Completeness). *For any Γ and φ, $\Gamma \vDash^D \varphi$ entails $\Gamma \vdash \varphi$.*

Proof. By Lemma 7, $\Gamma \vDash^D \varphi$ entails $\Gamma^D \vDash^D \varphi^D$. We may now apply Corollary 6 to obtain $\Gamma^D \vdash \varphi^D$ which can easily be shown to entail $\Gamma \vdash \varphi$ (for example, we do so in [6]). ◀

Observe that Theorem 8 was obtained fully constructively. This is noteworthy because similar results often make use of unconstructive principles. For example, the only method of extending completeness from the $\forall, \rightarrow, \bot$-fragment to the full syntax for classical Tarski semantics that we know of uses LEM [6]. This should be taken as an indication that material dialogues are exceptionally well-suited as a semantics for classical first-order logic in a constructive setting.

4 Intuitionistic Material Dialogues

Lorenzen's intuitionistic material dialogues differ from their classical counterparts by imposing one more restriction on the proponent: They may only ever defend against the opponent's most recent attack. This is analogous to the restriction to at most one right-hand formula in the intuitionistic sequent calculus. The proponent's possible moves are thus as given below. Note that in this section, in a slight abuse of notation, $\mathrm{Win}\,(\rho, A, C)$ and $\Gamma \vDash^D \varphi$ refer to dialogues played according to these intuitionistic rules.

$$\text{PA}\frac{\varphi \in A \quad a \rhd \varphi}{(\rho, A, C) \rightsquigarrow_p (\rho, A, C)\,; PA\, a} \qquad \text{PD}\frac{d \in \mathcal{D}_c \quad \rho \text{ justifies } d}{(\rho, A, c :: C) \rightsquigarrow_p d^P\,(\rho, A, c :: C)\,; \text{move}\, d}$$

Intuitionistic material dialogues do not admit a constructive completeness proof. To demonstrate this, we define the following fragment of first-order logic:

$$a, b : \mathtt{A} ::= \bot \mid P\,\vec{t} \mid a \wedge b \mid a \vee b \mid \exists a \qquad\qquad P : \Sigma, \vec{t} : \mathtt{T}^{|P|}$$
$$\varphi, \psi : \mathtt{F}^{\mathtt{D}} ::= a \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid a \rightarrow \psi \mid \forall x.\varphi \mid \exists x.\varphi$$

$\mathtt{A}$ is the fragment of $\mathtt{F}$ which allows attacking "blindly", meaning the same attack pattern can be used on these formulas in every winning strategy. The fragment $\mathtt{A}$ does not include $a \rightarrow b$ as attacking it requires being able to defend a and $\forall a$ as attacking it requires a (finite) choice of $s : \mathbf{S}$.

Unless specified otherwise, we are working in a standard structure $\mathbf{S}$.

▶ **Lemma 9.** *For any $a : \mathtt{A}$ and any ρ, A, C with $a \in A$ one may assume $\rho \vDash a$ to deduce $\mathit{Win}\,(\rho, A, C)$, that is, $(\rho \vDash a \rightarrow \mathit{Win}\,(\rho, A, C)) \rightarrow \mathit{Win}\,(\rho, A, C)$.*

Proof. Per induction on the structure of a. Suppose $\rho \vDash a$ entails $\mathrm{Win}\,(\rho, a :: A, C)$ we show $\mathrm{Win}\,(\rho, a :: A, C)$. Where appropriate, we implicitly make use of the fact that $\mathrm{Win}\,(\rho, A, C)$ entails $\mathrm{Win}\,(\vec{s} \cdot \rho, A', \uparrow^n C)$ where $\vec{s} : \mathbf{S}$, $|\vec{s}| = n$ and $\uparrow^n A \subseteq A'$.

$a = P\,\vec{t}$: Then the proponent attacks $P\,\vec{t}$, forcing the opponent to admit $\rho \vDash P\,\vec{t}$. The proponent may then continue according to the assumption.

$a = \bot$: The proponent attacks $\bot$ and wins.

$a = a \wedge b$: The proponent starts by attacking $a \wedge b$ with A_L and A_R, leaving us to prove that $\mathrm{Win}\,(\rho, a :: b :: A, C)$. Applying the IH for a and b means we may assume $\rho \vDash a$ and $\rho \vDash b$ to prove $\mathrm{Win}\,(\rho, a :: b :: A, C)$. As we thus know $\rho \vDash a \wedge b$ the proponent can proceed per assumption.

$a = a \vee b$: The proponent attacks $a \vee b$, leaving $\mathrm{Win}\,(\rho, c :: a \vee c :: A, C)$ for $c \in a, b$. Applying the IH for c allows us to assume $\rho \vDash c$, meaning the proponent can continue per assumption in either case.

$a = \exists a$: The proponent attacks $\exists a$, leaving $\mathrm{Win}\,(s \cdot \rho, a :: \uparrow(\exists a :: A), \uparrow C)$. Per IH on a we may assume $s \cdot \rho \vDash a$ and continue per assumption. ◀

▶ **Theorem 10.** *Pick $\varphi : \mathtt{F}^{\mathtt{D}}$, then $\rho \vDash \varphi$ entails $\mathit{Win}\,(\rho, A, C, \varphi)$ for any A and C.*

Proof. Proof per induction on φ.

$\varphi = a$ **:** We only handle $\varphi = P\vec{t}$ and $\varphi = \bot$ as the other cases are subsumed by other cases of this proof. If $\rho \vDash \bot$ we are done. If $\rho \vDash P\vec{t}$ then the only possible challenge is $A_P\,\vec{t}$ to which the proponent can respond by admitting $\rho \vDash P\vec{t}$.

$\varphi = \varphi \wedge \psi$ **:** Then we know $\rho \vDash \varphi$ and $\rho \vDash \psi$. The possible challenges are A_L and A_R, defending against which leaves $\mathrm{Win}\,(\rho, A, A_X :: C, \theta)$ for some $\theta \in \{\varphi, \psi\}$. Either case holds per IH for θ.

$\varphi = \varphi \vee \psi$ **:** Then we know $\rho \vDash \theta$ for $\theta \in \{\varphi, \psi\}$. The proponent thus defends against $A_\vee$ by admitting θ and proceeds per IH for θ.

$\varphi = a \to \psi$ **:** Then we know $\rho \vDash a$ entails $\rho \vDash \psi$. In attacking, the opponent will admit a, leaving $\mathrm{Win}\,(\rho, a :: A, A_\to\, a\,\psi :: C)$. We apply Lemma 9, allowing us to assume $\rho \vDash a$ to prove $\mathrm{Win}\,(\rho, a :: A, A_\to\, a\,\psi :: C)$. The proponent thus defends by admitting ψ and proceeds per IH on ψ as $\rho \vDash \psi$ per assumption.

$\varphi = \forall\varphi$ **:** We know that $s \cdot \rho \vDash \varphi$ for any $s : \mathbf{S}$. The challenge will be $A_s\,\varphi$ for some $s : \mathbf{S}$. The proponent thus reacts by admitting φ, proceeding per IH.

$\varphi = \exists\varphi$ **:** Then $s \cdot \rho \vDash \varphi$ for some $s : \mathbf{S}$. The only possible challenge is $A_\exists\,\varphi$ to which the proponent responds by admitting φ with s as the witness, proceeding per IH. ◀

Note that $\mathtt{F}^{\mathtt{D}}$ can be simplified, up to intuitionistic equivalence, by taking $\mathtt{A}$ to be only $P\,\vec{t}$ and $\bot$. We opted to demonstrate the result for the more complex fragment as it makes it more apparent why we chose exactly this fragment (the "blind attack" justification).

Theorem 10 means that if the meta-logic is at least as strong as some non-intuitionistic intermediate logics, their axiom schemata for formulas for $a, b : \mathtt{A}$ are valid in intuitionistic dialogues. Some examples of such axiom schemata are:

- Classical logic C: $a \vee \neg a$
- Gödel-Dummett logic LC: $(a \to b) \vee (b \to a)$
- Logics of bounded cardinality BC_n: $\bigvee_{i=1}^n \bigwedge_{j<i} a_j \to a_i$
- Logics of bounded width BW_n: $\bigvee_{i=1}^n \bigwedge_{j\neq i} a_j \to a_i$
- Logics of bounded depth BD_n: $a_n \vee (a_n \to (a_{n-1} \vee (a_{n-1} \to ...(a_2 \vee (a_2 \to (a_1 \vee \neg a_1))))))$

As these intermediate logics are consistent with the CIC, there is no hope of proving the completeness of intuitionistic material dialogues over the CIC without additional assumptions as this would contradict the aforementioned consistencies. However, there might be such a proof under axioms guaranteeing the CIC to behave truly intuitionistically.

If we assume the full law of the excluded middle, we can obtain an even stronger result: intuitionistic and classical dialogical validity fully coincide. This result relies on the following lemma:

▶ **Lemma 11.** *Assuming LEM, the following holds for any φ in any standard structure*

1. $\forall \rho, A, C.\ \rho \vDash \neg\varphi \to \varphi \in A \to \mathit{Win}\,(\rho, A, C)$
2. $\forall \rho, A, C.\ \rho \vDash \varphi \to \mathit{Win}\,(\rho, A, C, \varphi)$

Proof. We show both claims simultaneously per induction on φ. For most cases, 2. works the same as in Theorem 10 in which case we omit them.

$\varphi = P\,\vec{t}$ **: 1.** The proponent may force the opponent to demonstrate $\rho \vDash P\,\vec{t}$ by attacking $P\,\vec{t} \in A$, contradicting $\rho \vDash \neg P\,\vec{t}$.

$\varphi = \bot$ **: 1.** The proponent may win by attacking $\bot \in A$.

$\varphi = \varphi \to \psi$ **:**

1. Suppose $\rho \vDash \neg(\varphi \rightarrow \psi)$, meaning $\rho \vDash \varphi$ and $\rho \vDash \neg\psi$. The proponent then attacks $\varphi \rightarrow \psi \in A$. If the opponent counters the attack, the proponent can win by playing the strategy obtained by IH2 on $\rho \vDash \varphi$. If the opponent admits ψ, then the proponent plays according to IH1 on $\rho \vDash \neg\psi$.
2. Suppose $\rho \vDash \varphi \rightarrow \psi$. The opponent attacks $\varphi \rightarrow \psi$ with $A_{\rightarrow}\,\varphi\,\psi$, admitting φ. By the law of the excluded middle, either $\rho \vDash \varphi$ or $\rho \vDash \neg\varphi$. In the latter case, the proponent can now proceed per IH2 on $\rho \vDash \neg\varphi$. In the former case we have $\rho \vDash \psi$ per assumption and the proponent can proceed by admitting ψ and playing along IH1 on $\rho \vDash \psi$.

$\varphi = \varphi \wedge \psi$ **: 1.** Suppose $\rho \vDash \neg(\varphi \wedge \psi)$, meaning $\rho \vDash \neg\varphi$ or $\rho \vDash \neg\psi$. The proponent thus attacks the side of the contradicted formula of $\varphi \wedge \psi \in A$ and proceeds per IH1.

$\varphi = \varphi \vee \psi$ **: 1.** Suppose $\rho \vDash \neg(\varphi \vee \psi)$, meaning $\rho \vDash \neg\varphi$ and $\rho \vDash \neg\psi$. By attacking $\varphi \vee \psi \in A$, the proponent thus forces the opponent to admit either clause, thus being able to proceed via IH1.

$\varphi = \forall\varphi$ **:** If $\rho \vDash \neg\forall\varphi$ that means there is an $s : \mathbf{S}$ with $s \cdot \rho \vDash \neg\varphi$. The proponent thus attack $\forall\varphi$ with $A_s\,\varphi$ and proceeds per IH1.

$\varphi = \exists\varphi$ **:** Suppose $\rho \vDash \neg\exists\varphi$, meaning $s \cdot \rho \vDash \neg\varphi$ for any $s : \mathbf{S}$. Then the proponent attacks $\exists\varphi \in A$ and proceeds per IH1. ◀

▶ **Corollary 12.** *Under the LEM, classical and intuitionistic dialogical validity coincide.*

Proof.

$\leftarrow$: This is the case even without the law of excluded middle as every winning strategy for an intuitionistic material dialogue is a winning strategy for the classical material dialogues on the same state.

$\rightarrow$: Suppose $\Gamma \vDash^D \varphi$ classically. In Lemma 5 we have shown that this means $\Gamma \vDash^E \varphi$. As every standard structure is exploding and under the LEM every structure is classical, this means φ is valid under Γ in every standard structure. By Lemma 11 this means that $\Gamma \vDash^D \varphi$ intuitionistically. ◀

Lastly, it should be noted that this failure of completeness is not simply due to the wrong choice of rules for material intuitionistic dialogues. Indeed, formal dialogues for intuitionistic first-order logic, which differ from the dialogues investigated in this section only by their treatment of atomic formulas, are constructively sound and complete for the full syntax of first-order logic [4, 5]. This failure of completeness is thus solely due to the material nature of material intuitionistic dialogues.

5 Material Kripke Dialogues

The previous section demonstrates that intuitionistic material dialogues fail to capture intuitionistic first-order logic. In this section, we give an alternative dialogical semantics which succeeds in this. Classical material dialogues can be seen as classical dialogues played on Tarski models, the canonical notion of model for classical first-order logic. In that vein, we present intuitionistic dialogues played on Kripke models as a semantics for intuitionistic first-order logic. These stray far from the ideas of Lorenzen but are nonetheless interesting in their own right.

A material Kripke dialogue is played on a Kripke structure $K, \leq, \iota$. The game states of material Kripke dialogues are dependent pairs $(k, \rho, A, C) : \Sigma k : K,\ \mathbf{V} \rightarrow \mathbf{S}_k \times \mathcal{L}(\mathtt{F}) \times \mathcal{L}(\mathbf{A})$ which can be viewed as material dialogue states "at a world k in K". To mirror the clauses of $\Vdash$ for $\rightarrow$ and $\forall$, opponent attacks in Kripke dialogues can move the game state along $\leq$ in K. To express this we define a predicate $a|k \mapsto k'$ with where $A_{\rightarrow}\,\varphi\,\psi|k \mapsto k'$ and $A_s\,\varphi|k \mapsto k'$

hold whenever $k \leq k'$ and $a|k \mapsto k$ holds for all other attacks a. The definitions of valid moves and their effects on the game state are largely analogous to intuitionistic material dialogues, this time also incorporating the movement along the Kripke frame. Note that the definition of d^P and d^O are essentially the same as for the previous material dialogues, leaving the world unchanged.

$$\text{PA}\,\frac{\varphi \in A \quad a \rhd \varphi}{(k, \rho, A, C) \rightsquigarrow_p (k, \rho^k, A, C)\,;\, PA\, a}$$

$$\text{PD}\,\frac{d \in \mathcal{D}_c \quad \rho \text{ justifies } d}{(k, \rho, A, c :: C) \rightsquigarrow_p d^P\,(k, \rho, A, c :: C)\,;\,\text{move}\, d}$$

$$\text{OA}\,\frac{c \rhd \varphi \quad c|k \mapsto k'}{(k, \rho, A, C)\,;\, PD\,\varphi \rightsquigarrow_o (k', \rho^{k'}, \psi :: A, c :: C)}$$

$$\text{OC}\,\frac{a \rhd \varphi \quad \mathsf{adm}\, a = \ulcorner \psi \urcorner \quad \psi \rhd c \quad c|k \mapsto k'}{(k, \rho, A, C)\,;\, PA\, a \rightsquigarrow_o (k', \rho^{k'}, c :: A, c :: C)}$$

$$\text{OD}\,\frac{d \in \mathcal{D}_a \quad \rho \text{ justifies } d}{(k, \rho, A, C)\,;\, PA\, a \rightsquigarrow_o d^O\,(k, \rho, A, C)}$$

The definition of $\text{Win}\,(k, \rho, A, C)$ essentially remains unchanged. We modify the definition of $\text{Win}\,(k, \rho, A, C, \varphi)$ to be that we have $\text{Win}\,(k', \rho, a :: A, c :: C)$ for any $a \rhd \varphi$ with $a|k \mapsto k'$. We then define $\Gamma \Vdash^D \varphi$ the same as before.

5.1 Soundness

We prove that material Kripke dialogues are sound with regards to the cut-free intuitionistic sequent calculus given in Appendix B. The proof is somewhat simpler than that for classical material dialogues because of the restriction of the proponent being only able to defend against the opponent's most recent challenge and the analogous restriction to only one right-hand formula in the sequent calculus. We once again require the congruences laid out in Appendix C as captured by Lemma 13.

▶ **Lemma 13.** *Suppose $Win\,(k, \rho, A, C)$ and $\rho, A \equiv_f \rho', A'$, $\rho, C \equiv_a \rho', C'$ for some ρ', A' and C'. Then $Win\,(k, \rho', A', C')$.*

▶ **Theorem 14.** *Suppose $\Gamma \Rightarrow \delta$ then $\Gamma \Vdash^D \delta$.*

Proof. For this, we prove $\Gamma \Rightarrow \delta \to \forall k, \rho, C.\ \text{Win}\,(k, \rho, \Gamma, \delta)$ per induction on $\Gamma \Rightarrow \delta$. We again only handle two exemplary cases.

L$\to$: The proponent thus attacks $\varphi \to \psi \in \Gamma$. If the opponent counters, the proponent plays according to the IH on $\Gamma \Rightarrow \varphi$. If the opponent defends, the proponent plays according to the IH on $\Gamma, \psi \Rightarrow \delta$.

R$\forall$: Then the challenge is $A_s\,\varphi$ for some $s : \mathbf{S}_{k'}$ and $A_s\,\varphi|k \mapsto k'$. The proponent can then defend and continue playing according to the IH on $\uparrow\!\Gamma \Rightarrow \varphi$.

◀

5.2 Completeness

We show completeness restricted to the $\forall, \rightarrow, \bot$-fragment the same way we do in Section 3.2.

We begin by showing that dialogical validity entails exploding Kripke validity.

▶ **Lemma 15.** *Whenever* $\Gamma \Vdash^D \varphi$ *then also* $\Gamma \Vdash^E \varphi$.

Proof. Fix an exploding Kripke structure $K, \leq, \iota$. We extend the forcing relation to defenses

$$\rho^k \Vdash D_A\, \varphi \Leftrightarrow \rho^k \Vdash \varphi \qquad \rho^k \Vdash D_W\, \varphi\, s \Leftrightarrow s \cdot \rho^k \Vdash \varphi \qquad \rho^k \Vdash D_M\, \varphi \Leftrightarrow \rho^k \Vdash \varphi$$

and define an auxiliary predicate

$$\Gamma \Vdash^k_\rho \bigvee \mathcal{D}_c \Leftrightarrow \forall k \leq k', \alpha.\ \rho^{k'} \Vdash \Gamma \rightarrow (\forall d \in \mathcal{D}_c.\ \rho^{k'} \Vdash d \rightarrow \rho^{k'} \Vdash \alpha) \rightarrow \rho^{k'} \Vdash \alpha$$

We again make use of two results, proven in Appendix E:

1. If for some φ we have $(c :: \Gamma) \Vdash^k_\rho \bigvee \mathcal{D}_c$ for all $c \rhd \varphi$ then $\rho^k \Vdash \Gamma$ entails $\rho^k \Vdash \varphi$.
2. If $\mathrm{Win}\,(k, \rho, \Gamma, c :: C)$ then $\Gamma \Vdash^k_\rho \bigvee \mathcal{D}_c$.

Then $\rho^k \Vdash \varphi$ can be concluded from $\rho^K \Vdash \Gamma$ analogously to Corollary 6. ◀

▶ **Theorem 16** (Completeness)**.** *Restricting to the* $\forall, \rightarrow, \bot$*-fragment,* $\Gamma \Vdash^D \varphi$ *entails* $\Gamma \vdash^I \varphi$.

Proof. By Lemma 15 we know $\Gamma \Vdash^E \varphi$. By Theorem 1 we may conclude $\Gamma \vdash^I \varphi$. ◀

In contrast to the classical material dialogues, completeness for material Kripke dialogues cannot be extended to the full syntax by using a DeMorgan translation, as those are only applicable in classical settings. Indeed, we do not know of a constructive method which would allow for this extension. However, Veldman [19] has given a proof that the fan theorem implies completeness for exploding Kripke models on the full syntax, which could be applied to obtain full completeness for material Kripke dialogues, albeit unconstructively.

6 Discussion

Mechanization of active research While working on this paper, we mechanized all results from Section 3, safe for some of the corollaries, in the interactive theorem prover Coq. Mechanizing already established results in Coq is a worthy endeavor in its own right, for example yielding some insight into their computational contents when working without additional non-constructive axioms. However, we want to discuss using Coq to mechanize new results while working on them as we did here. Mechanizing the results of Section 3 revealed some mistakes in our initial definition of the rules for material dialogues which, albeit being minor, invalidated both soundness and completeness. We missed these mistakes while working out the proofs on paper and believe it would have taken us much longer to spot them without the mechanization. Having machine checked our definitions in Section 3 gave us sufficient confidence to work on paper for the remainder of the project. It should also be noted that the mechanization took up only about a quarter of the overall time spent on the project, thanks in part due to building on top of the a large preexisting mechanization from [6]. We feel this might be a worthwhile trade-off between the time requirement of a full mechanization of all results and the room for error in working solely on paper.

Proof strategies for Completeness In this paper, we prove completeness by relating dialogical validity to validity in some model-based semantics and then using preexisting completeness. The general reasoning was that this is the quickest way to obtain these completeness theorems in the framework set up by [6]. For classical material dialogues, we believe it would also be possible to obtain a direct constructive completeness proof with regards to natural deduction on the basis of a Henkin construction, although it would likely require the dialogical cut. For material Kripke dialogues, we believe we could obtain a direct constructive completeness proof for the $\forall, \rightarrow, \bot$-fragment via a normalization-by-evaluation approach, similar to that by Herbelin and Lee [9]. However, we regard material Kripke validity entailing exploding Kripke validity on the full syntax as a stronger result because of its broader scope, which is why we opted for the proof strategy exhibited in this report.

Classical Kripke Dialogues The analysis of Kripke dialogues for intuitionistic first-order logic naturally brings up the question of how Kripke dialogues with a classical rule set would behave. While we chose not to pursue this question further, we conjecture that Kripke dialogues with classical rules should behave the same as material dialogues with classical rules. Their validity should entail classical exploding Tarski validity, as every classical exploding Tarski structure can also be viewed as an equivalent one-world Kripke structure. The more critical property is soundness: We believe that the "independence of the classicality of the structure" already demonstrated by the soundness proof for classical material dialogues should extend to even work on Kripke structures. We have, however, not checked this formally.

Benefits of Material Dialogues Another question naturally raised by the results of this paper is how material dialogues compare to other semantics for first-order logic, especially in a constructive setting. As outlined in Section 2, when working with Tarski semantics in the CIC, one's attention needs to be restricted to classical structures. As an example, the standard model for Peano arithmetic is not provably classical and can thus not be studied. In contrast, Section 3 demonstrates that classical material dialogues embody classical logic regardless of the classicality of the underlying structure. It thus seems like a promising basis on which to carry out model-theoretic investigations of classical first-order logic in constructive settings. Specifically of interest seems the question whether it might allow deriving consistency and underivability results for classical first-order theories without having to rely on non-constructive axioms to obtain models such as in [11]. However, we have not yet investigated these possibilities more deeply. For material Kripke dialogues, on the other hand, there seems no apparent benefit over simply working with Kripke structures.

Faithfulness to Lorenzen's material dialogues We have tried to be as faithful to Lorenzen's definitions from [15, 16] as possible while also implementing the idea of material dialogues played over first-order structures. Arguably, this is already in contradiction to Lorenzen's ideas as he placed a lot of value in the "underlying game" for settling atomic propositions to be of a discrete nature, something completely lost in our formulation. However, all the attacks and defenses for the connectives of first-order logic are exactly as they are in Lorenzen's work. Coincidentally, the idea of using a structure defined constant $\bot^{\mathbf{S}}$ is very similar to Lorenzen's definitions which propose to fix some "unwinnable" proposition as a stand-in for $\bot$. Of course, the modifications made when moving to material Kripke dialogues are completely outside of the spirit of his ideas.

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A DeBruijn Indexes

DeBruijn indexes [3] were developed by Nicolaas Govert de Bruijn as part of the implementation of the AUTOMATH theorem prover [2]. They provide a formalism for treating syntax containing binders and substitutions with greater ease than the more common "named binders" approach. Given below is the syntax of first-order logic using DeBruijn binders. Note especially the absence of variable names after the quantifiers.

$$t : \mathtt{T} ::= n \mid c\,\vec{t} \qquad\qquad n : \mathbb{N}, c : \Sigma$$
$$\varphi : \mathtt{F} ::= \bot \mid P\,\vec{t} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \forall\varphi \mid \exists\varphi \qquad\qquad P : \Sigma$$

In DeBruijn syntax, the variables are represented by natural numbers $n : \mathbb{N}$, called DeBruijn indexes. Such an index references the quantifier binding it by counting the number of quantifiers above it in the syntax tree that need to be skipped to arrive at the binding quantifier. For example, the formula $\forall x.P\,x \to \forall z.Q\,x\,z$ becomes $\forall\ P\,0 \to \forall\ Q\,1\,0$. Figure 3 gives a visual representation of the references described by the indexes. Observe that the variable x is represented by different indexes depending on the position of its occurrence in the syntax tree.

$$\forall \quad P\;0 \quad \to \quad \forall \quad Q\;1\;0$$

Figure 3 A DeBruijn formula

When working with DeBruijn binders, most definitions involving logical formulas need to be adjusted slightly. For example, environments become maps $\rho : \mathbb{N} \to \mathbf{S}$ mapping indexes to elements of the structure. The usual environment extension operation $\rho[x \mapsto s]$ is replaced by the operation $s \cdot \rho$ which is defined as below

$$(s \cdot \rho)(n) := \begin{cases} s & n = 0 \\ \rho(m) & n = m + 1 \end{cases}$$

The quantifier rules for Tarski semantics then change appropriately as follows:

$$\rho \vDash \forall\varphi :\Leftrightarrow \forall s : \mathbf{S},\ \ s \cdot \rho \vDash \varphi \qquad\qquad \rho \vDash \exists\varphi :\Leftrightarrow \exists s : \mathbf{S},\ \ s \cdot \rho \vDash \varphi$$

Note that below the quantifier, the indexes of all variables not bound by that quantifier are increased by 1 thus ensuring that each variable refers to the correct element of $s \cdot \rho$. The rules for Kripke semantics are adjusted in a similar manner.

The definitions of the dialogue games also change when using DeBruijn binders. For instance, the attacks and defenses with witnesses do not include the name of the binding variable anymore because it is always identified by the index 0.

$$A_s\,\varphi \rhd \forall\varphi \qquad \mathcal{D}_{A_s\,\varphi} = \{D_W\,\varphi\,s\}$$
$$A_\exists\,\varphi \rhd \exists x.\varphi \qquad \mathcal{D}_{A_\exists\,\varphi} = \{D_W\,\varphi\,s \mid s : \mathbf{S}\}$$

Furthermore, the operations d^P and d^O are changed accordingly. Here, $\uparrow\!\varphi$ is the operation which increases the index of each free variable occurring in φ by 1, thus freeing the index 0 for the newly introduced $s : \mathbf{S}$. This operation is extended to attacks and lists of attacks or formulas in the obvious way.

$$d^P\,(\rho, A, C) = \begin{cases} (s \cdot \rho, \uparrow\!A, \uparrow\!C) & \text{if } d = D_W\,\varphi\,s \\ (\rho, A, C) & \text{otherwise} \end{cases}$$

$$d^O\,(\rho, A, C) = \begin{cases} (\rho, \varphi :: A, C) & \text{if } d = D_A\,\varphi \\ (s \cdot \rho, \varphi ::\uparrow A, \uparrow C) & \text{if } (D_W\,\varphi\, s) \\ (\rho, A, C) & \text{if } d = D_M\,\varphi \end{cases}$$

The definitions of d^P and d^O for material Kripke dialogues are analogous.

B Derivation Systems

We define two cut-free sequent calculi used in Sections 3.1 and 5.1, respectively. The sequent calculi are somewhat non-standard, expressing the formula requirements of the various rules in terms of list membership in the sequents Γ and Δ. This is convenient when proving soundness as it parallels the requirements on attacking and defending moves for the proponent.

The *classical sequent calculus* is defined as follows:

$$\text{Ax}\frac{P\,\vec{t} \in \Gamma \quad P\,\vec{t} \in \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{L}\bot\frac{\bot \in \Gamma}{\Gamma \Rightarrow \Delta} \qquad \text{L}{\rightarrow}\frac{\varphi \rightarrow \psi \in \Gamma \quad \Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{R}{\rightarrow}\frac{\varphi \rightarrow \psi \in \Delta \quad \Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{L}\wedge\frac{\varphi \wedge \psi \in \Gamma \quad \Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{R}\wedge\frac{\varphi \wedge \psi \in \Delta \quad \Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{L}\vee\frac{\varphi \vee \psi \in \Gamma \quad \Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{R}\vee\frac{\varphi \vee \psi \in \Delta \quad \Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{L}\forall\frac{\forall\varphi \in \Gamma \quad \Gamma, \varphi[t] \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{R}\forall\frac{\forall\varphi \in \Delta \quad \uparrow\Gamma \Rightarrow \varphi, \uparrow\Delta}{\Gamma \Rightarrow \Delta} \qquad \text{L}\exists\frac{\exists\varphi \in \Gamma \quad \uparrow\Gamma, \varphi \Rightarrow \uparrow\Delta}{\Gamma \Rightarrow \Delta} \qquad \text{R}\exists\frac{\exists\varphi \in \Delta \quad \Gamma \Rightarrow \varphi[t], \Delta}{\Gamma \Rightarrow \Delta}$$

The intuitionistic sequent calculus is given below:

$$\text{Ax}\frac{P\,\vec{t} \in \Gamma}{\Gamma \Rightarrow P\,\vec{t}} \qquad \text{L}\bot\frac{\bot \in \Gamma}{\Gamma \Rightarrow \delta} \qquad \text{L}{\rightarrow}\frac{\varphi \rightarrow \psi \in \Gamma \quad \Gamma \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow \delta}{\Gamma \Rightarrow \delta}$$

$$\text{R}{\rightarrow}\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} \qquad \text{L}\wedge\frac{\varphi \wedge \psi \in \Gamma \quad \Gamma, \varphi, \psi \Rightarrow \delta}{\Gamma \Rightarrow \delta} \qquad \text{R}\wedge\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$$

$$\text{L}\vee\frac{\varphi \vee \psi \in \Gamma \quad \Gamma, \varphi \Rightarrow \delta \quad \Gamma, \psi \Rightarrow \delta}{\Gamma \Rightarrow \delta} \qquad \text{R}\vee\text{L}\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad \text{R}\vee\text{R}\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}$$

$$\text{L}\forall\frac{\forall\varphi \in \Gamma \quad \Gamma, \varphi[t] \Rightarrow \delta}{\Gamma \Rightarrow \delta} \qquad \text{R}\forall\frac{\uparrow\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \forall\varphi} \qquad \text{L}\exists\frac{\exists\varphi \in \Gamma \quad \uparrow\Gamma, \varphi \Rightarrow \uparrow\delta}{\Gamma \Rightarrow \delta}$$

$$\text{R}\exists\frac{\Gamma \Rightarrow \varphi[t]}{\Gamma \Rightarrow \exists\varphi}$$

C Dialogue congruences

Given ρ, ρ' and φ, φ', we define an equivalence relation $\rho, \varphi \equiv_f \rho', \varphi'$ as below. Intuitively, $\rho, \varphi \equiv_f \rho', \varphi'$ means that φ and φ' are equal up to their terms and the pairs of terms at a certain position within these formulas are equal under evaluation in the respective environment.

$$\frac{}{\rho, \bot \equiv_f \rho', \bot} \qquad \frac{\vec{t}^{\rho} = \vec{t'}^{\rho'}}{\rho, P\,\vec{t} \equiv_f \rho', P\,\vec{t'}} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi' \quad \rho, \psi \equiv_f \rho', \psi'}{\rho, \varphi \Box \psi \equiv_f \rho', \varphi' \Box \psi'}$$

$$\frac{\forall d.\ (d \cdot \rho), \varphi \equiv_f (d \cdot \rho'), \varphi'}{\rho, \Box\varphi \equiv_f \rho', \Box\varphi'}$$

We then extend this congruence to attacks $\rho, a \equiv_a \rho', a'$, defenses $\rho, d \equiv_d \rho', d'$.

$$\frac{}{\rho, A_\bot \equiv_a \rho', A_\bot} \qquad \frac{\vec{t}^{\rho} = \vec{t'}^{\rho'}}{\rho, A_P\,\vec{t} \equiv_a \rho', A_P\,\vec{t'}} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi'}{\rho, A_L\,\varphi \equiv_a \rho', A_L\,\varphi'}$$

$$\frac{\rho, \psi \equiv_f \rho', \psi'}{\rho, A_R\,\psi \equiv_a \rho', A_R\,\psi'} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi' \quad \rho, \psi \equiv_f \rho', \psi'}{\rho, A_\vee\,\varphi\,\psi \equiv_a \rho', A_\vee\,\varphi'\,\psi'}$$

$$\frac{\rho, \varphi \equiv_f \rho', \varphi' \quad \rho, \psi \equiv_f \rho', \psi'}{\rho, A_\to\,\varphi\,\psi \equiv_a \rho', A_\to\,\varphi'\,\psi'} \qquad \frac{(s \cdot \rho), \varphi \equiv_f (s \cdot \rho'), \varphi'}{\rho, A_s\,\varphi \equiv_a \rho', A_s\,\varphi'}$$

$$\frac{\forall d.\ (d \cdot \rho), \varphi \equiv_f (d \cdot \rho'), \varphi'}{\rho, A_\exists\,\varphi \equiv_a \rho', A_\exists\,\varphi'}$$

$$\frac{\vec{t}^{\rho} = \vec{t'}^{\rho'}}{\rho, D_M\,P\,\vec{t} \equiv_d \rho', D_M\,P\,\vec{t'}} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi'}{\rho, D_A\,\varphi \equiv_d \rho', D_A\,\varphi'} \qquad \frac{(s \cdot \rho), \varphi \equiv_f (s \cdot \rho'), \varphi'}{\rho, D_W\,\varphi\,s \equiv_d \rho', D_W\,\varphi'\,s}$$

We extend all of the above to lists with

$$\frac{}{\rho, [] \equiv_x \rho', []} \qquad \frac{\rho, a \equiv_x \rho', a' \quad \rho, A \equiv_x \rho', A'}{\rho, (a :: A) \equiv_x \rho', (a' :: A')}$$

Lastly we define a congruence between substitutions $\rho, \sigma \equiv_s \rho', \sigma'$ which holds if for all variables x we have that $(\sigma\,x)^{\rho} = (\sigma'\,x)^{\rho'}$. We now state all of the properties required of the relations to show Lemma 3. All of them have been proven in the Coq mechanization accompanying this report.

▶ **Proposition 17.**
1. *$\rho, \varphi \equiv_f \rho', \varphi'$ is an equivalence relation*
2. *$\rho, \varphi \equiv_a \rho', \varphi'$ is an equivalence relation*
3. *$\rho, \varphi \equiv_d \rho', \varphi'$ is an equivalence relation*
4. *$\rho, \varphi \equiv_f \rho', \varphi'$ and $a \rhd \varphi$ mean there is a $a' \rhd \varphi'$ with $\rho, a \equiv_a \rho', a'$*
5. *If $\rho, a \equiv_a \rho', a'$ then if $\mathsf{adm}\, a = \ulcorner \varphi \urcorner$ then $\mathsf{adm}\, a' = \ulcorner \varphi' \urcorner$ such that $\rho, \varphi \equiv_f \rho', \varphi'$*
6. *$\rho, a \equiv_a \rho', a'$ and $d \in \mathcal{D}_a$ mean there is a $d' \in \mathcal{D}_{a'}$ with $\rho, d \equiv_d \rho', d'$*
7. *If $\rho, d \equiv_d \rho', d'$ and ρ justifies d then ρ' justifies d'*
8. *If $\rho, a \equiv_a \rho', a'$ and $\rho, A \equiv_f \rho', A'$ then $\rho, (a :: A) \equiv_f \rho', (a' :: A')$*
9. *$\rho, \sigma \equiv_s \rho', \sigma'$ means $\rho, \varphi[\sigma] \equiv_f \rho', \varphi[\sigma']$*
10. *$\rho, \sigma \equiv_s \rho', \sigma'$ means $\rho, a[\sigma] \equiv_a \rho', a[\sigma']$*

D Proofs for Classical Material Dialogues

D.1 Fragment Completeness

For this, we extend the Tarski satisfaction relation to defenses.

$$\rho \vDash D_A\,\varphi \Leftrightarrow \rho \vDash \varphi \qquad \rho \vDash D_W\,\varphi(x)\,s \Leftrightarrow \rho[x \mapsto s] \vDash \varphi \qquad \rho \vDash D_M\,\varphi \Leftrightarrow \rho \vDash \varphi$$

We define an auxiliary predicate on contexts Γ, environments ρ and sets of defenses Δ

$$\Gamma \vDash_\rho \bigvee \Delta \Leftrightarrow \rho \vDash \Gamma \to \forall \vec{s} : \mathbf{S}, \alpha.\ (\forall d \in \Delta.\ \rho \vDash d \to (\vec{s} \cdot \rho) \vDash \alpha) \to (\vec{s} \cdot \rho) \vDash \alpha$$

As the notation suggests, $\Gamma \vDash_\rho \bigvee \Delta$ is an impredicative formalization of the claim that under the environment ρ, Γ entails the disjunction of all semantical interpretations of Δ.

▶ **Lemma 18.** *Pick some classical, exploding structure $\mathbf{S}$, environment ρ, context Γ, set Δ and formula φ. If for all $c \rhd \varphi$ we have $(c :: \Gamma) \vDash_\rho \bigvee(\mathcal{D}_c \cup \Delta)$ then $\Gamma \vDash_\rho \bigvee(\{D_A\,\varphi\} \cup \Delta)$.*

Proof. We assume (1) $\forall c \rhd \varphi.\ (c :: \Gamma) \vDash_\rho \bigvee(\mathcal{D}_c \cup \Delta)$. To show $\Gamma \vDash_\rho \bigvee\{D_A\,\varphi\} \cup \Delta$ we assume (2) $\forall d \in \{D_A\,\varphi\} \cup \Delta.\ \rho \vDash d \to (\vec{s} \cdot \rho) \vDash \alpha$ and $\rho \vDash \Gamma$ to deduce $(\vec{s} \cdot \rho) \vDash \alpha$. The proof proceeds per case distinction on φ. We only prove some exemplary cases.

$\varphi = \varphi \to \psi$ **:** As $\mathbf{S}$ is classical, we may apply Peirce's law and assume $(\vec{s} \cdot \rho) \vDash \neg\alpha$. Per (2), it suffices to show $\rho \vDash \varphi \to \psi$. We thus assume $\rho \vDash \varphi$ and apply (1) for $A_\to\,\varphi\,\psi$, yielding $(\varphi :: \Gamma) \vDash_\rho \bigvee\{D_A\,\psi\} \cup \Delta$, to deduce $\rho \vDash \psi$. For the case $d = D_A\,\psi$ this is trivial, thus suppose $d \in \Delta$. By (2), $\rho \vDash d$ entails $(\vec{s} \cdot \rho) \vDash \alpha$ and thus $\bot^{\mathbf{S}}$, meaning $\rho \vDash \psi$ as $\mathbf{S}$ is exploding.

$\varphi = \varphi \vee \psi$ **:** Applying (1) for $A_\vee\,\varphi\,\psi$ together with (2) directly yields the claim. ◀

Note that the ability to extend the context in $\Gamma \vDash_\rho \Delta$ is used in the $\forall\varphi$ case of the proof above. Indeed, this is the only where we make use of it overall.

▶ **Lemma 19.** *Let $\mathbf{S}$ be classical and exploding and (ρ, A, C) a state. If $Win(\rho, A, C)$ then $A \vDash_\rho \bigvee \mathcal{D}_C$ where $\mathcal{D}_C = \bigcup_{c \in C} \mathcal{D}_c$.*

Proof. We proceed per induction on $\mathrm{Win}\,(\rho, A, C)$ and perform a case distinction on the proponent move. We assume (H) $\forall d \in \mathcal{D}_C.\ \rho \vDash d \to (\vec{s} \cdot \rho) \vDash \alpha$ and $\rho \vDash A$. We only handle some of the cases.

***PA a* :** Then there is some $\varphi \in A$ with $a \rhd \varphi$. We perform a case distinction on φ.

- $\varphi = \varphi \to \psi$ **:** Then the IH upon the opponent countering together with Lemma 18 yields (1) $A \vDash_\rho \{D_A\,\varphi\} \cup \mathcal{D}_C$ and upon the opponent defending is (2) $(\psi :: A) \vDash_\rho \mathcal{D}_C$. We first assume $(\vec{s} \cdot \rho) \vDash \neg\alpha$ by Peirce's law and then apply (2). This leaves us proving $\rho \vDash \psi$ which we can do by proving $\rho \vDash \varphi$ as $\varphi \to \psi \in A$. We conclude this from (1), (H) and $(\vec{s} \cdot \rho) \vDash \neg\alpha$.

***PD* φ :** Then defending results in a state (ρ', A', C') and the IH together with Lemma 18 yields $A' \vDash_{\rho'} \{D_A\,\varphi\} \cup \mathcal{D}_{C'}$. We further know that there is a $c \in C$ and a defense $d \in \mathcal{D}_c$ such that $\rho \vDash d \Leftrightarrow \rho' \vDash \varphi$. We may thus apply the IH to resolve the claim as $D_A\,\varphi$ does not "add anything" to $\mathcal{D}_{C'}$ and the validity of (H) is maintained under the possible transformations applied to (ρ, A, C).

***PM* φ :** Then $\rho \vDash \varphi$ holds and there is a $c \in C$ with $c \rhd \varphi$. Then we may apply (H) with $P_M\,\varphi \in \mathcal{D}_C$ to deduce $(\vec{s} \cdot \rho) \vDash \alpha$. ◀

D.2 Dialogical Cut-Elimination

We say a formula φ *can be cut* if for any ρ, A, A', C we have that $\mathrm{Win}\,(\rho, A +\!\!+ \varphi :: A', C)$ and $\mathrm{Win}\,(\rho, A +\!\!+ A', C, \varphi)$ entails $\mathrm{Win}\,(\rho, A +\!\!+ A', C)$. The proof proceeds in two steps.

The proofs in this section heavily rely on the weakening principles Lemmas 3 and 20. However, we feel that spelling out all applications of these principles obscures the simple ideas behind this section's proofs. We thus opt to leave applications of Lemmas 3 and 20 implicit where possible. Readers interested in the proofs in full detail may take a look at the Coq mechanization accompanying this report.

▶ **Lemma 20** (Weakening). *Let $Win(\rho, A, C)$ and $A \subseteq A'$, $C \subseteq C'$ then $Win(\rho, A', C')$.*

Proof. Simple induction on $\mathrm{Win}(\rho, A, C)$. ◀

▶ **Lemma 21.** *Pick a formula φ such that all formulas of smaller complexity can be cut. Now pick some n and $c \rhd (\uparrow^n \varphi)$ such that $Win(\rho, A, C +\!\!+ c :: C')$ and for all $d \in \mathcal{D}_c$ justified under ρ we have $Win(d^O(\rho, A, C +\!\!+ C'))$. Then $Win(\rho, A, C +\!\!+ C')$.*

Proof. We proceed per induction on $\mathrm{Win}(\rho, A, C +\!\!+ c :: C')$. We first perform a case distinction on the proponent's move in $\mathrm{Win}(\rho, A, C +\!\!+ c :: C')$.

PA: The proponent uses $a \rhd \psi$ on some $\psi \in A$. Then the proponent of $\mathrm{Win}(\rho, A, C +\!\!+ C')$ copies that move. There are two possible opponent responses.
- In the case of $\mathsf{adm}\, a = \ulcorner\theta\urcorner$, the opponent may counter with some $c' \rhd \theta c'$. Then the proponent copies the strategy obtained from the inductive hypothesis upon the same counter.
- The opponent may defend with some $d \in \mathcal{D}_a$. Then the proponent copies the strategy obtained for the inductive hypothesis upon the same defense.

PD: Then there is a $c' \in C +\!\!+ c :: C'$ and the proponent defends with some $d \in \mathcal{D}_c$. There are two cases to distinguish:

$c' \in C +\!\!+ C'$: Then the proponent of $\mathrm{Win}(\rho, A, C +\!\!+ C')$ copies the defense. If d is not $D_M\, P\, \vec{t}$ for some $P, \vec{t}$, then the opponent attacks the formula ψ admitted by d with some $a \rhd \psi$. The proponent then plays according to the strategy obtained from the inductive hypothesis upon $a \rhd \psi$.

$c' = c$: Per assumption we have $\mathrm{Win}(d^O(\rho, A, C +\!\!+ C'))$. Then we perform a case distinction on the form of d.

$d = D_M\, \varphi$: Then $\mathrm{Win}(d^O(\rho, A, C +\!\!+ C')) = \mathrm{Win}(\rho, A, C +\!\!+ C')$ and we are done.

$d = D_A\, \psi$: The assumption thus is $\mathrm{Win}(\rho, \psi :: A, C +\!\!+ C')$. From the inductive hypothesis we obtain $\mathrm{Win}(\rho, A, C +\!\!+ C', \psi)$. As $D_A\, \psi \in \mathcal{D}_c$ and $c \rhd (\uparrow^n \varphi)$ we know that ψ is of lower complexity than φ, meaning it can be cut and we thus obtain $\mathrm{Win}(\rho, A, C +\!\!+ C')$.

$d = D_W\, \psi\, s$: This case is analogous to that for $d = D_A\, \psi$ with a few more applications of Lemma 3.

◀

▶ **Theorem 22** (Dialogical Cut). *All formulas can be cut.*

Proof. The proof proceeds per induction on formula complexity. Thus pick a φ such that all formulas of lower complexity can be cut. We show that

$$\mathrm{Win}(\rho, A +\!\!+ \uparrow^n \varphi :: A', C) \to \mathrm{Win}(\rho, A +\!\!+ A', C, \uparrow^n \varphi) \to \mathrm{Win}(\rho, A +\!\!+ A', C)$$

per induction on $\mathrm{Win}(\rho, A +\!\!+ \uparrow^n \varphi :: A', C)$ which subsumes the fact that φ can be cut. We perform a case distinction on the proponent move.

PA a Then the proponent attacks some $\psi \in A +\!\!+ \uparrow^n \varphi :: A'$ with $a \rhd \psi$. We distinguish two cases.

$\psi \in A +\!\!+ A'$: Then the proponent of $\mathrm{Win}(\rho, A +\!\!+ A', C)$ copies that attack and proceeds per inductive hypothesis.

$\psi = \uparrow^n \varphi$ **:** Then $\mathrm{Win}\,(\rho, A +\!\!+ A', C, \uparrow^n \varphi)$ yields $\mathrm{Win}\,(\rho, A +\!\!+ A', a :: C)$ and the inductive hypothesis means that for all $d \in \mathcal{D}_a$ we have that $\mathrm{Win}\,(d^O\,(\rho, A +\!\!+ A', C))$. We may thus apply Lemma 21 to deduce $\mathrm{Win}\,(\rho, A +\!\!+ A', C)$.

$PD\,\psi$ **:** Then there is some $c \in C$ and some $d \in \mathcal{D}_c$ such that d results in admitting ψ. The proponent of $\mathrm{Win}\,(\rho, A +\!\!+ A', C)$ thus copies that admission and proceeds per inductive hypothesis.

$PM\,\varphi$ **:** Then $c \in C$ for the unique $a \rhd \varphi$ (in either case of $\varphi = \bot$ or $\varphi = P\,\vec{t}$) and $\rho \vDash \varphi$ holds. The proponent of $\mathrm{Win}\,(\rho, A +\!\!+ A', C)$ can thus win as well by demonstrating $\rho \vDash \varphi$. ◀

E Proofs for Material Kripke Dialogues

Fix an exploding Kripke structure $K, \leq, \iota$. We extend the forcing relation to defenses

$$\rho^k \Vdash D_A\,\varphi \Leftrightarrow \rho^k \Vdash \varphi \qquad \rho^k \Vdash D_W\,\varphi\,s \Leftrightarrow s \cdot \rho^k \Vdash \varphi \qquad \rho^k \Vdash D_M\,\varphi \Leftrightarrow \rho^k \Vdash \varphi$$

and define an auxiliary predicate

$$\Gamma \Vdash^k_\rho \bigvee \mathcal{D}_c \Leftrightarrow \forall k \leq k', \alpha.\ \rho^{k'} \Vdash \Gamma \to (\forall d \in \mathcal{D}_c.\ \rho^{k'} \Vdash d \to \rho^{k'} \Vdash \alpha) \to \rho^{k'} \Vdash \alpha$$

▶ **Lemma 23.** *If for some φ we have $(c :: \Gamma) \Vdash^k_\rho \bigvee \mathcal{D}_c$ for all $c \rhd \varphi$ then $\rho^k \Vdash \Gamma$ entails $\rho^k \Vdash \varphi$.*

Proof. Assume $(\star)$ $\forall c \rhd \varphi.\ (c :: \Gamma) \Vdash^k_\rho \bigvee \mathcal{D}_c$ and $\rho^k \Vdash \Gamma$. Proceed per case distinction on φ.

$\varphi = \bot$ **:** As there are no defenses against $A_\bot$, choosing $\bot$ for α in $(\star)$ already yields $\rho^k \Vdash \bot$.

$\varphi = P\,\vec{t}$**:** We apply $(\star)$ to $A_P\,\vec{t}$.

$\varphi = \varphi \to \psi$ **:** Let $k \leq k'$ and suppose $\rho^{k'} \Vdash \varphi$. As this means $\rho^{k'} \Vdash \varphi :: \Gamma$, we may apply $(\star)$ with $A_\to\,\varphi\,\psi$.

$\varphi = \varphi \wedge \psi$ **:** Then we can apply $(\star)$ with $A_L\,\varphi$ and $A_R\,\psi$ to obtain $\rho^k \Vdash \varphi$ and $\rho^k \Vdash \psi$, yielding $\rho^k \Vdash \varphi \wedge \psi$ overall.

$\varphi = \varphi \vee \psi$ **:** Then we apply $(\star)$ with $A_\vee\,\varphi\,\psi$. This leaves us proving that $\rho^k \Vdash \varphi$ and $\rho^k \Vdash \psi$ each entail $\rho^k \Vdash \varphi \vee \psi$ which is clear.

$\varphi = \forall\varphi$ **:** Let $k \leq k'$ and $s : \mathbf{S}_{k'}$, we need to prove that $s \cdot \rho^{k'} \Vdash \varphi$. For this, we apply $(\star)$ with $A_s\,\varphi$.

$\varphi = \exists\varphi$ **:** For this we apply $(\star)$ with $A_\exists\,\varphi$. This leaves us proving that for $s : \mathbf{S}_k$, $s \cdot \rho^k \Vdash \varphi$ entails $\rho^k \Vdash \exists\varphi$ which is clear. ◀

▶ **Lemma 24.** *Let $K, \leq, \iota$ be exploding. Suppose $\mathit{Win}\,(k, \rho, \Gamma, c :: C)$ then $\Gamma \Vdash^k_\rho \bigvee \mathcal{D}_c$.*

Proof. We proceed per induction on $\mathrm{Win}\,(k, \rho, \Gamma, c :: C)$. Suppose $k \leq k'$ and $\rho^{k'} \Vdash \Gamma$. Now suppose that $\rho^{k'} \Vdash d$ entails $\rho^{k'} \Vdash \alpha$ for all $d \in \mathcal{D}_c$.

PA **:** The proponent attacks some $\varphi \in \Gamma$. We perform a case distinction on Γ.

- $\varphi = \bot$ **:** Then $\rho^{k'} \Vdash \bot$ per assumption and thus $\rho^{k'} \Vdash \alpha$ as the structure is exploding.
- $\varphi = P\,\vec{t}$**:** Then we may apply the IH for the opponent defending by demonstrating $\rho^{k'} \Vdash P\,\vec{t}$ which holds per assumption.
- $\varphi = \varphi \to \psi$ **:** We first apply Lemma 23 to the IH upon the opponent countering to obtain $\rho^{k'} \Vdash \varphi$. We can then apply the IH obtained upon the opponent admitting ψ as $\rho^{k'} \Vdash \varphi :: \Gamma$.
- $\varphi = \varphi \wedge \psi$ **:** To apply the IH we have to demonstrate $\rho^k \Vdash \varphi$ or $\rho^k \Vdash \psi$, either of which hold as $\rho^k \Vdash \varphi \wedge \psi$.

$\varphi = \varphi \vee \psi$ **:** Then either $\rho^{k'} \Vdash \varphi$ or $\rho^{k'} \Vdash \psi$. In either case, we can apply the IH upon the opponent admitting the respective formula.

$\varphi = \forall \varphi$ **:** The proponent chooses some $s : \mathbf{S}_k$. As $\rho^{k'} \Vdash \forall \varphi$ we have $s \cdot \rho^{k'} \Vdash \varphi$. By applying the IH upon the opponent admitting this, we can then obtain $s \cdot \rho^{k'} \Vdash \uparrow \alpha$ meaning $\rho^{k'} \Vdash \alpha$.

$\varphi = \varphi \vee \psi$ **:** Then either $\rho^{k'} \Vdash \varphi$ or $\rho^{k'} \Vdash \psi$. In either case, we can apply the IH upon the opponent admitting the respective formula. with $s \cdot \rho^{k'} \Vdash \varphi$. The proof then proceeds analogously to the case of $\varphi = \forall \varphi$.

$\boldsymbol{PD}$ **:** The proponent thus defends against c via some $d \in \mathcal{D}_c$. We thus show $\rho^{k'} \Vdash \alpha$ by showing $\rho^k \Vdash d$. If $d = P_M\, P\, \vec{t}$ that means the proponent demonstrated $\rho^k \Vdash P\, \vec{t}$ which is transported to $\rho^{k'} \Vdash P\, \vec{t}$ via ι, meaning $\rho^{k'} \Vdash d$. In the other two cases, we may apply Lemma 23 to the IH to obtain $\rho^{k'} \Vdash d$.

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