Material Dialogues for First-Order Logic in Constructive Type Theory*

Dominik Wehr[1,3] and Dominik Kirst[2]

[1] University of Gothenburg, Sweden
[2] Saarland University, Saarland Informatics Campus, Germany
[3] University of Amsterdam, The Netherlands

Abstract. Material dialogues are turn taking games which model debates about the satisfaction of logical formulas. A novel variant played over first-order structures gives rise to a notion of first-order satisfaction. We study the induced notion of validity for classical and intuitionistic first-order logic in the constructive setting of the calculus of inductive constructions. We prove that such material dialogue semantics for classical first-order logic admits constructive soundness and completeness proofs, setting it apart from standard model theoretic semantics of first-order logic. Furthermore, we prove that completeness with regards to intuitionistic material dialogues fails in constructive and classical settings. The results concerning classical material dialogues have been mechanized using the Coq interactive theorem prover.

Keywords: Dialogue Games · Game Semantics · First-Order Logic · Constructive Type Theory

1 Introduction

Logical dialogues were introduced by Paul Lorenzen [12, 13], a philosopher and constructive mathematician active throughout the latter half of the twentieth century. They are a result of his search for a constructively acceptable account of mathematics, beginning with his work on operative mathematics [11]. Logical dialogues are turn-taking games which model a debate in which the proponent defends the validity of a formula against the criticisms of an opponent. The games' moves are modeled after speech acts: asserting formulas and questioning assertions made by the other player. In this, they differ from the more widespread logical games in the style of Hintikka [8], in which a formula is reduced to its atoms by both players, the turn order being determined by the syntactical structure of the formula. Although logical dialogues were initially put forward as a semantics for intuitionistic logic, they can also capture classical logic [10].

Any dialogue begins with the proponent asserting a formula to be discussed. The players then take turns, starting with the opponent. The player at turn can

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choose between two possible moves: Either they can attack an assertion made by the opposing player or they can defend against such an attack, usually by asserting another formula. Some attacks require the attacking player to assert a formula while carrying out the attack, in which case the assertion in question is called an admission. As an example, attacking the asserted formula $\varphi \to \psi$ requires the attacker to admit φ while attacking. To defend against the attack on $\varphi \to \psi$, the attacked player must assert ψ. As in many games, finite plays are won by the player who made the last move. Infinite plays are always won by the opponent. To prevent the opponent from winning by stalling indefinitely, for example through repeated attacks on the proponent's initial assertion, additional restrictions are imposed on the opponent's legal moves. The most common such restriction is to only allow the opponent to react to the proponent's previous move (see Krabbe [9] for alternative restrictions).

While the meaning of logical connectives can be captured by attacks and defenses allowing the players to break down assertions into subformulas, this approach does not extend to atomic formulas. There are two different ways of incorporating atomic formulas in logical dialogues. Material dialogues, the variant originally proposed by Lorenzen [12, 13], permit attacks on atomic formulas. To defend against such an attack, the attacked player is required to demonstrate the validity of the atomic formula. In Lorenzen's original formulation this meant deriving a word according to a grammar, a remainder of his operative semantics of mathematics [11]. Formal dialogues, which were put forward in the dissertation of Lorenzen's student Kuno Lorenz [10], treat atomic formulas without appealing to their "underlying meaning", i.e. by purely syntactic means. In this setting, atomic formulas cannot be attacked by either player and an additional restriction is imposed on the proponent: They may only assert those atomic formulas which the opponent has asserted previously. Historically, the study of logical dialogues after Lorenzen's and Lorenz' initial work has been focused on formal dialogues due to their greater simplicity [9]. Sørensen and Urzyczyn [15] have demonstrated that the winning strategies of formal dialogues for propositional logic are structurally similar to sequent calculus derivations, a result which has been extended to first-order logic in [4].

If one fixes the demonstration method of atomic formulas in a material dialogue to be a demonstration of its satisfaction in a previously agreed upon first-order structure, this induces a model-theoretic notion of satisfaction and, by quantifying over all models, validity. In this article, we study the arising semantics of first-order logic in the constructive setting of the calculus of inductive constructions [2, 14]. This extends our previous investigation [4, 5] into the constructivity of completeness theorems for various semantics of first-order logic, including formal dialogues for intuitionistic first-order logic.

1.1 Outline and Contributions

This section summarizes the article's results. Section 2 covers some basic definitions and results we rely on throughout the article. We close with a brief discussion of various questions arising from this article in Section 5. Note that our use of "soundness" and "completeness" may be somewhat unusual. When speaking of the soundness of a semantics, the intended meaning is the soundness of some suitable deduction system with regards to said semantics (and similarly for "completeness"). While non-standard, this terminology allows us to be more concise about the results derived in this article.

Classical material dialogues In Section 3, we define material dialogues for classical first-order logic with all connectives. We prove their soundness with regards to a cut-free, classical sequent calculus. Notably, classical material dialogues are sound on any first-order structure, whereas classical Tarski semantics require the underlying structure to satisfy all instances of the law of the excluded middle (LEM), a property not necessarily held by all structures in a constructive setting. One could thus say that the "classicality" of classical material dialogues is within their rules of engagement, not the underlying structures. We further prove that classical material validity entails exploding classical Tarski validity, a constructively stricter notion than standard classical Tarski validity. We then use the constructive completeness of exploding classical Tarski validity to deduce the same for classical material dialogues. Notably, we obtain completeness for the full syntax of first-order logic. The only analogous result for Tarski semantics we are aware of relies on the full LEM [5]. The results of Section 3 have been mechanized in Coq, the mechanization being located at [17].

Intuitionistic material dialogues In Section 4 we analyze material dialogues for first-order logic with the usual rules of dialogues of intuitionistic logics. We prove that standard Tarski validity on the fragment $\mathfrak{F}^{\mathsf{D}}$ given below entails intuitionistic material validity.

$$
\begin{aligned}
a, b : \mathfrak{A} &::= \bot \mid P\,\bar{t} \mid a \land b \mid a \lor b \mid \exists x.a && P : \Sigma, \bar{t} : \mathfrak{T}, x : \mathbf{V} \\
\varphi, \psi : \mathfrak{F}^{\mathsf{D}} &::= a \mid \varphi \land \psi \mid \varphi \lor \psi \mid a \to \psi \mid \forall x.\varphi \mid \exists x.\varphi && x : \mathbf{V}
\end{aligned}
$$

This means that proving completeness with regards to intuitionistic material dialogues is tantamount to disproving non-constructive principles on the meta-level. As such principles are consistent with most constructive theories, completeness cannot be established without additional axioms. In fact, intuitionistic and classical material dialogues completely coincide under the full LEM. The standard rendition of intuitionistic material dialogues is thus ill-suited as a semantics of intuitionistic first-order logic.

2 Preliminaries

2.1 The Calculus of Inductive Constructions

The results of this article are all derived within the Calculus of Inductive Constructions (CIC) [2, 14], the type theory underlying the interactive theorem prover Coq. The CIC consists of a predicative hierarchy of type universes $\mathbb{T}_i$ above an impredicative universe $\mathbb{P}$ of propositions. Each type universe contains an empty type, products $A \times B$, sums $A + B$, function types $A \to B$, dependent products $\Pi a : A.B(a)$ and dependent sums $\Sigma a : A.B(a)$. In $\mathbb{P}$, we denote them by their respective Curry-Howard correspondents $\bot, \land, \lor, \to, \forall, \exists$. Allowing unrestricted elimination from $\mathbb{P}$ into the $\mathbb{T}_i$ results in an inconsistency [6]. However, this restriction can be lifted for some types in $\mathbb{P}$, including types of at most one constructor, such as $\bot$ and the equality type $= : \Pi A.A \to A \to \mathbb{P}$ with a sole constructor of type $\forall (a : A).\ a = a$.

We use inductive types for the natural numbers ($n : \mathbb{N} := 0 \mid S\,n$), option types ($\mathcal{O}(A) := \ulcorner a \urcorner \mid \emptyset$) and list types ($l : \mathcal{L}(A) := [] \mid a :: l$). We denote *list membership* by $a \in l$ and the *list appending operation* by $l \mathbin{+\!\!+} l'$.

2.2 First-Order Predicate Logic

Fix a *signature* Σ of functions symbols f and predicate symbols P, denoting their arities by $|f|$ and $|P|$, respectively. Let $\mathbf{V}$ be the countable type of variables $x, y, z : \mathbf{V}$. The associated *term and formula language* is defined below.

$$\mathfrak{T} ::= x \mid f\,\vec{t} \qquad x : \mathbf{V}, f : \Sigma, \vec{t} : \mathfrak{T}^{|f|}$$

$$\varphi : \mathfrak{F} ::= \dot{\bot} \mid P\,\vec{t} \mid \varphi \mathbin{\dot\land} \psi \mid \varphi \mathbin{\dot\lor} \psi \mid \varphi \mathbin{\dot\to} \psi \mid \dot\forall x.\varphi \mid \dot\exists x.\varphi \qquad x : \mathbf{V}, P : \Sigma, \vec{t} : \mathfrak{T}^{|P|}$$

To aid in the distinction between meta- and object-level syntax, we write small dots over the connectives of the latter. Negation is defined as $\dot\neg\,\varphi := \varphi \mathbin{\dot\to} \dot\bot$. Note that formally, especially within proofs, we are working with de Bruijn binders [1] instead of the syntax with named binders we present here. More details on de Bruijn binders are given in Section 2.3. However, for the sake of readability, we opt to present all definitions and theorems in the main text of this article in the familiar style of named binders.

A *structure* $\mathbf{S}$ consists of a type X, a *predicate interpretation* $P^{\mathbf{S}} : X^{|P|} \to \mathbb{P}$ for each $P : \Sigma$, a *function interpretation* $f^{\mathbf{S}} : X^{|f|} \to X$ for each $f : \Sigma$ and an *absurdity interpretation* $\dot\bot^{\mathbf{S}} : \mathbb{P}$. A *model* consists of a structure $\mathbf{S}$ together with an *assignment* $\rho : \mathbf{V} \to X$. The term evaluation function t^ρ in a model $\mathbf{S}, \rho$ is defined as $x^\rho := \rho\,x$ and $(f\,\vec{t})^\rho := f^{\mathbf{S}}\,\vec{t}^\rho$. The Tarski satisfaction relation $\rho \vDash \varphi$ is defined below. We often write $\mathbf{S}$ for X, e.g. writing $s : \mathbf{S}$ instead of $s : X$.

$$\begin{aligned} \rho \vDash P\,\vec{t} &:\Leftrightarrow P^{\mathbf{S}}\,\vec{t}^\rho & \rho \vDash \varphi \mathbin{\dot\to} \psi &:\Leftrightarrow \rho \vDash \varphi \to \rho \vDash \psi \\ \rho \vDash \varphi \mathbin{\dot\land} \psi &:\Leftrightarrow \rho \vDash \varphi \land \rho \vDash \psi & \rho \vDash \varphi \mathbin{\dot\lor} \psi &:\Leftrightarrow \rho \vDash \varphi \lor \rho \vDash \psi \\ \rho \vDash \dot\forall x.\varphi &:\Leftrightarrow \forall s : \mathbf{S},\ \rho[x \mapsto s] \vDash \varphi & \rho \vDash \dot\exists x.\varphi &:\Leftrightarrow \exists s : \mathbf{S},\ \rho[x \mapsto s] \vDash \varphi \\ \rho \vDash \dot\bot &:\Leftrightarrow \dot\bot^{\mathbf{S}} \end{aligned}$$

For a finite context Γ we write $\rho \vDash \Gamma$ if $\rho \vDash \varphi$ for all $\varphi \in \Gamma$. A structure $\mathbf{S}$ is *classical* if for all assignments ρ and formulas φ it satisfies the principle of double-negation elimination ($\rho \vDash \dot\neg\,\dot\neg\,\varphi \mathbin{\dot\to} \varphi$). A structure $\mathbf{S}$ is *exploding* if for all assignments ρ and formulas φ it satisfies the principle of explosion ($\rho \vDash \dot\bot \mathbin{\dot\to} \varphi$). A structure is $\mathbf{S}$ *standard* if $\dot\bot^{\mathbf{S}}$ is contradictory (i.e. $\neg\dot\bot^{\mathbf{S}}$ holds). Observe that all standard structures are exploding. An entailment between a finite context

Γ and a formula φ is *valid in classical exploding structures*, written $\Gamma \vDash^E \varphi$, if for all classical, exploding structures **S** and all assignments ρ it is the case that $\rho \vDash \Gamma$ entails $\rho \vDash \varphi$. *Validity of entailments in classical standard structures*, $\Gamma \vDash^S \varphi$, is defined analogously.

Let $\Gamma \Rightarrow \Delta$ denote derivability in the cut-free sequent calculus for classical first-order logic defined below. The system's presentation is somewhat non-standard as the principal formulas of derivation rules are not removed from Γ and Δ. This is convenient when proving soundness as it parallels the requirements imposed on the proponent when attacking and defending.

$$\text{Ax}\,\frac{P\,\vec{t} \in \Gamma \quad P\,\vec{t} \in \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{L}\bot\,\frac{\dot{\bot} \in \Gamma}{\Gamma \Rightarrow \Delta}$$

$$\text{L}\!\rightarrow\frac{\varphi \dot{\rightarrow} \psi \in \Gamma \quad \Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{R}\!\rightarrow\frac{\varphi \dot{\rightarrow} \psi \in \Delta \quad \Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{L}\wedge\frac{\varphi \dot{\wedge} \psi \in \Gamma \quad \Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{R}\wedge\frac{\varphi \dot{\wedge} \psi \in \Delta \quad \Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{L}\vee\frac{\varphi \dot{\vee} \psi \in \Gamma \quad \Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{R}\vee\frac{\varphi \dot{\vee} \psi \in \Delta \quad \Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{L}\forall\frac{\dot{\forall} \varphi \in \Gamma \quad \Gamma, \varphi[t] \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{R}\forall\frac{\dot{\forall} \varphi \in \Delta \quad \uparrow\Gamma \Rightarrow \varphi, \uparrow\Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{L}\exists\frac{\dot{\exists} \varphi \in \Gamma \quad \uparrow\Gamma, \varphi \Rightarrow \uparrow\Delta}{\Gamma \Rightarrow \Delta} \qquad \text{R}\exists\frac{\dot{\exists} \varphi \in \Delta \quad \Gamma \Rightarrow \varphi[t], \Delta}{\Gamma \Rightarrow \Delta}$$

The following completeness result stems from [7] based on ideas from [16]:

Theorem 1. *In constructive settings, restricting to the* $\dot{\forall}, \dot{\rightarrow}, \dot{\bot}$*-fragment of first-order logic,* $\Gamma \vDash^E \varphi$ *entails* $\Gamma \Rightarrow \varphi$.

2.3 De Bruijn binders

De Bruijn binders were developed by de Bruijn as part of the implementation of the AUTOMATH theorem prover [1]. They provide a formalism for treating syntax containing binders and substitutions with greater ease than the common "named binders" approach. Given below is the syntax of first-order logic using de Bruijn binders. Note especially the absence of variable names after the quantifiers.

$$t : \mathfrak{T} ::= n \mid c\,\vec{t} \qquad\qquad n : \mathbb{N}, c : \Sigma$$

$$\varphi : \mathfrak{F} ::= \dot{\bot} \mid P\,\vec{t} \mid \varphi \dot{\wedge} \psi \mid \varphi \dot{\vee} \psi \mid \varphi \dot{\rightarrow} \psi \mid \dot{\forall} \varphi \mid \dot{\exists} \varphi \qquad\qquad P : \Sigma$$

In de Bruijn syntax, the variables are represented by natural numbers $n : \mathbb{N}$, called de Bruijn indexes. Such an index references the quantifier binding it by counting the number of quantifiers above it in the syntax tree that need to be skipped to arrive at the binding quantifier. For an example consider the formula $\dot\forall x(P\,x \mathrel{\dot\to} \dot\forall z.Q\,x\,z)$ which is represented by $\dot\forall(P\,0 \mathrel{\dot\to} \dot\forall Q\,1\,0)$ in de Bruijn syntax. Figure 1 depicts the references described by the indexes. Observe that the variable x is represented by different indexes depending on its position in the syntax tree.

$$\dot\forall \quad P\,0 \quad \dot\to \quad \dot\forall \quad Q\,1\,0$$

Fig. 1. De Bruijn representation of $\forall x.P\,x \to \forall z.Q\,x\,z$

When working with de Bruijn binders, most definitions involving logical formulas need to be adjusted slightly. For example, assignments become functions $\rho : \mathbb{N} \to \mathbf{S}$ mapping indexes to elements of the structure. The usual assignment extension operation $\rho[x \mapsto s]$ is replaced by the operation $s \cdot \rho$ which is defined as below

$$(s \cdot \rho)(n) := \begin{cases} s & n = 0 \\ \rho(m) & n = m + 1 \end{cases}$$

The quantifier rules for Tarski semantics then change appropriately as follows:

$$\rho \vDash \dot\forall \varphi :\Leftrightarrow \forall s : \mathbf{S},\ s \cdot \rho \vDash \varphi \qquad\qquad \rho \vDash \dot\exists \varphi :\Leftrightarrow \exists s : \mathbf{S},\ s \cdot \rho \vDash \varphi$$

Note that below the quantifier, the indexes of all variables not bound by that quantifier are incremented thus ensuring that each variable refers to the correct element in $s \cdot \rho$.

3 Classical Material Dialogues

We begin by defining formally the material dialogues for classical first-order logic. Fix a standard structure $\mathbf{S}$ the dialogue will be played over. Material dialogues are a turn taking game between two players. The *proponent* tries to defend the satisfaction of some formula in a model, whereas the *opponent* tries to challenge the proponent's claims in such a way that the proponent cannot respond. The dialogues we consider are so-called *E-dialogues* which restrict the opponent to only ever react to the proponent's previous move. It can be shown that the notion of satisfaction induced by E-dialogues is equivalent to that of the more intuitive D-dialogues, in which this restriction is weakened [3, 5].

We model the dialogue game as a state transition system. A triple $(\rho, A, C) : (\mathbf{V} \to \mathbf{S}) \times \mathcal{L}(\mathfrak{F}) \times \mathcal{L}(\mathbf{A})$ is called a *dialogue state*. Together $\mathbf{S}, \rho$ form the *ambient model*. The list A contains all of the opponent's *assertions* and C

records all attacks that the opponent has leveled against the proponent. Each round begins with the proponent making a move, indicated by a transition $(\rho, A, C) \rightsquigarrow_p (\rho', A', C'); m$ from a dialogue state to a dialogue state and a proponent move $m : \mathbf{M}$. This is followed by an opponent move, indicated by a transition $(\rho', A', C'); m \rightsquigarrow_o (\rho'', A'', C'')$ from a state and proponent move to a further state. We continue by defining the two transition relations $\rightsquigarrow_p$ and $\rightsquigarrow_o$.

The *type* $\boldsymbol{D}$ *of defenses*, defined below, features three different kinds of defenses: $D_A\,\varphi$ denotes the act admitting of the formula φ, $D_W\,\varphi(x)\,s$ denotes admitting $\varphi(s)$ where $s : \mathbf{S}$. Lastly, $D_M\,\varphi$ means claiming to be able to demonstrate that φ holds. Note that $D_M\,\varphi$ is only ever instantiated with atomic φ.

$$\mathbf{D} ::= D_A\,\varphi \mid D_W\,\varphi(x)\,s \mid D_M\,\varphi \qquad\qquad \varphi : \mathfrak{F}, s : \mathbf{S}$$

The type $\mathbf{A}$ contains all *attacks*. One writes $a \rhd \varphi$ if $a : \mathbf{A}$ is an *attack on* φ. Each $a : \mathbf{A}$ has an associated set $\mathcal{D}_a$ of *defenses against* a. Both $\mathbf{A}$ and the associated $\mathcal{D}_a$ are laid out below.

$$\begin{aligned}
A_\perp &\rhd \dot\perp & \mathcal{D}_{A_\perp} &= \{D_M\,\dot\perp\} \\
A_P\,\vec{t} &\rhd P\,\vec{t} & \mathcal{D}_{A_P\,\vec{t}} &= \{D_M\,P\,\vec{t}\} \\
A_\rightarrow\,\varphi\,\psi &\rhd \varphi \mathbin{\dot\rightarrow} \psi & \mathcal{D}_{A_\rightarrow\,\varphi\,\psi} &= \{D_A\,\psi\} \\
A_\vee\,\varphi\,\psi &\rhd \varphi \mathbin{\dot\vee} \psi & \mathcal{D}_{A_\vee\,\varphi\,\psi} &= \{D_A\,\varphi, D_A\,\psi\} \\
A_L\,\varphi &\rhd \varphi \mathbin{\dot\wedge} \psi & \mathcal{D}_{A_L\,\varphi} &= \{D_A\,\varphi\} \\
A_R\,\psi &\rhd \varphi \mathbin{\dot\wedge} \psi & \mathcal{D}_{A_R\,\psi} &= \{D_A\,\psi\} \\
A_s\,\varphi(x) &\rhd \dot\forall x.\varphi(x) & \mathcal{D}_{A_s\,\varphi(x)} &= \{D_W\,\varphi(x)\,s\} \\
A_\exists\,\varphi(x) &\rhd \dot\exists x.\varphi(x) & \mathcal{D}_{A_\exists\,\varphi(x)} &= \{D_W\,\varphi(x)\,s \mid s : \mathbf{S}\}
\end{aligned}$$

Some attacks force the attacker to *admit* a formula. This is formalized by a function $\mathsf{adm} : \mathbf{A} \to \mathcal{O}(\mathfrak{F})$ where $\mathsf{adm}\,a = \ulcorner\varphi\urcorner$ means that φ must be admitted when attacking with a and $\mathsf{adm}\,a = \emptyset$ means no admission need be made. The admission obligations are $\mathsf{adm}\,(A_\rightarrow\,\varphi\,\psi) = \ulcorner\varphi\urcorner$ and $\mathsf{adm}\,a = \emptyset$ for all other attacks $a : \mathbf{A}$.

Each round begins by the proponent making a *move* $m : \mathbf{M}$, as detailed below: Either challenging one of the opponent's assertions ($PA\,a$) or defending against a challenge previously issued by the opponent, either by asserting a formula ($PD\,\varphi$) or by demonstrating that an atomic formula holds in the ambient model ($PM\,\varphi$). The function $\mathsf{move} : \mathbf{D} \to \mathbf{M}$ maps defenses to the proponent move that needs to be made to carry out said defense.

$$m : \mathbf{M} := PA\,a \mid PD\,\varphi \mid PM\,\varphi$$

$$\mathsf{move}\,(D_A\,\varphi) = PD\,\varphi \qquad\qquad \mathsf{move}\,(D_W\,\varphi(x)\,s) = PD\,\varphi(x')$$

$$\mathsf{move}\,(D_M\,\varphi) = PM\,\varphi$$

The effect of a proponent's defense d on the game state is defined via a function d^P as below, where x' denotes a variable which does not occur in $A, C, \varphi(x)$.

$$d^P\,(\rho, A, C) = \begin{cases} (\rho[x' \mapsto s], A, C) & \text{if } d = D_W\,\varphi(x)\,s \\ (\rho, A, C) & \text{otherwise} \end{cases}$$

Defenses need to be *justified* by the ambient model. While $D_A\,\varphi$ and $D_W\,\varphi\,s$ are always justified, $D_M\,\varphi$ requires $\rho \vDash \varphi$ to hold. All of the previous notations enable a compact definition of the state transitions the proponent may induce by making a move.

$$\text{PA}\,\frac{\varphi \in A \quad a \rhd \varphi}{(\rho, A, C) \rightsquigarrow_p (\rho, A, C)\,;\, PA\,a} \qquad \text{PD}\,\frac{c \in C \quad d \in \mathcal{D}_c \quad \rho \text{ justifies } d}{(\rho, A, C) \rightsquigarrow_p d^P\,(\rho, A, C)\,;\, \mathsf{move}\,d}$$

The opponent must react to the proponent's previous move. If the proponent defended by asserting a formula, they must issue a new challenge by attacking said assertion (OA). If the proponent attacked one of their assertions, they can either defend against said attack (OD) or counter the attack, attacking the admission made by the proponent while attacking (OC). If the proponent demonstrated the validity of an atomic formula in the ambient model, the opponent cannot respond at all. The operation d^O is defined analogously to d^P and is used to define the transition steps the opponent can induce by making a move. In a slight abuse of notation, we write $c :: A$, where c is an attack, for $\psi :: A$ if $\mathsf{adm}\,c = \ulcorner\psi\urcorner$ and A if $\mathsf{adm}\,c = \emptyset$.

$$d^O\,(\rho, A, C) = \begin{cases} (\rho, \varphi :: A, C) & \text{if } d = D_A\,\varphi \\ (\rho[x' \mapsto s], \varphi(x') :: A, C) & \text{if } (D_W\,\varphi(x)\,s) \\ (\rho, A, C) & \text{if } d = D_M\,\varphi \end{cases}$$

$$\text{OA}\,\frac{c \rhd \varphi}{(\rho, A, C)\,;\, PD\,\varphi \rightsquigarrow_o (\rho, \psi :: A, c :: C)}$$

$$\text{OC}\,\frac{a \rhd \varphi \quad \mathsf{adm}\,a = \ulcorner\psi\urcorner \quad \psi \rhd c}{(\rho, A, C)\,;\, PA\,a \rightsquigarrow_o (\rho, c :: A, c :: C)}$$

$$\text{OD}\,\frac{d \in \mathcal{D}_a \quad \rho \text{ justifies } d}{(\rho, A, C)\,;\, PA\,a \rightsquigarrow_o d^O\,(\rho, A, C)}$$

A state can be *won* if the proponent can ensure the play always eventually ends. This is defined as an inductive predicate which is very similar to the inductive well-foundedness predicate commonly used in type theory. In the common parlance of game theory, each derivation of $\text{Win}\,s$ constitutes a winning strategy for the proponent on state s.

$$\frac{s \rightsquigarrow_p s'\,;\,m \qquad \forall s''.\ s'\,;\,m \rightsquigarrow_o s'' \to \text{Win}\,s''}{\text{Win}\,s}$$

This notion of winnability extends to formulas φ denoted $\mathrm{Win}\,(\rho, A, C, \varphi)$ which holds if for all attacks $c \rhd \varphi$ it is the case that $\mathrm{Win}\,(\rho, c :: A, c :: C)$. A formula φ is *dialogically entailed* in a context Γ, written $\Gamma \vDash^D \varphi$, if for all standard structures $\mathbf{S}$ and assignments $\rho : \mathbf{V} \to \mathbf{S}$ it is the case that $\mathrm{Win}\,(\rho, \Gamma, [], \varphi)$.

Example 1. Sketched below is a strategy establishing $\mathrm{Win}\,(\rho, [P \mathbin{\dot\to} Q, P], [A_Q])$ and thus $\mathrm{Win}\,(\rho, [P \mathbin{\dot\to} Q, P], [], Q)$. The proponent first attacks $P \mathbin{\dot\to} Q$. In both branches, the opponent can only continue if an atomic formula is satisfied by the ambient model. The proponent can thus carry out a "demonstration" (PM) of said atomic formula, winning the play. This strategy can be played on any model $\mathbf{S}, \rho$, meaning it suffices to establish $P \mathbin{\dot\to} Q, P \vDash^D Q$.

$$\begin{tikzcd}[column sep=tiny]
 & (\rho, [P \mathbin{\dot\to} Q, P], [A_Q]) \arrow[d, "\mathrm{PA}"'] & \\
 & (\rho, [P \mathbin{\dot\to} Q, P], [A_Q])\,;\, PA\,(A_\to\, P\, Q) \arrow[dl, "\mathrm{OC}"'] \arrow[dr, "\mathrm{OD}"] & \\
(\rho, [P \mathbin{\dot\to} Q, P], [A_Q, A_P]) \arrow[d, "\mathrm{PA}"'] & & (\rho, [P \mathbin{\dot\to} Q, P, Q], [A_Q]) \arrow[d, "\mathrm{PA}"'] \\
(\rho, [P \mathbin{\dot\to} Q, P], [A_Q, A_P])\,;\, PA\, A_P \arrow[d, "\mathrm{OD}"', "\text{requires } \rho \vDash P"] & & (\rho, [P \mathbin{\dot\to} Q, P, Q], [A_Q])\,;\, PA\, A_Q \arrow[d, "\mathrm{OD}"', "\text{requires } \rho \vDash Q"] \\
(\rho, [P \mathbin{\dot\to} Q, P], [A_Q, A_P]) \arrow[d, "\mathrm{PD}"', "\text{relies on } \rho \vDash P"] & & (\rho, [P \mathbin{\dot\to} Q, P, Q], [A_Q]) \arrow[d, "\mathrm{PD}"', "\text{relies on } \rho \vDash Q"] \\
(\rho, [P \mathbin{\dot\to} Q, P], [A_Q, A_P])\,;\, PM\, P & & (\rho, [P \mathbin{\dot\to} Q, P, Q], [A_Q])\,;\, PM\, Q
\end{tikzcd}$$

The definitions of dialogue games given above were in terms of named binders. When using de Bruijn binders, some definitions require slight adjustments. The attacks and defenses with witnesses need not include the name of the binding variable anymore because it is always identified by the index 0.

$$\begin{aligned} A_s\,\varphi \rhd \dot\forall\,\varphi \qquad & \mathcal{D}_{A_s\,\varphi} = \{D_W\,\varphi\, s\} \\ A_\exists\,\varphi \rhd \dot\exists\, x.\varphi \qquad & \mathcal{D}_{A_\exists\,\varphi} = \{D_W\,\varphi\, s \mid s : \mathbf{S}\} \end{aligned}$$

We denote by $\uparrow\!\varphi$ the operation which increments the index of each free variable occurring in φ, thus freeing the index 0. This operation can be extended to attacks and lists of attacks or formulas in the obvious way. The de Bruijn definitions of d^P and d^O employ this shifting operation to ensure that the index 0 refers to the newly introduced $s : \mathbf{S}$ in the case of $d = D_W\,\varphi\, s$.

$$d^P\,(\rho, A, C) = \begin{cases} (s \cdot \rho, \uparrow\! A, \uparrow\! C) & \text{if } d = D_W\,\varphi\, s \\ (\rho, A, C) & \text{otherwise} \end{cases}$$

$$d^O\,(\rho, A, C) = \begin{cases} (\rho, \varphi :: A, C) & \text{if } d = D_A\,\varphi \\ (s \cdot \rho, \varphi :: \uparrow\! A, \uparrow\! C) & \text{if } d = D_W\,\varphi\, s \\ (\rho, A, C) & \text{if } d = D_M\,\varphi \end{cases}$$

We continue by proving the classical material dialogues sound with regards to the cut-free classical sequent calculus from Section 2.2. This is the easiest soundness result to obtain because of the structural similarity between winning strategies for dialogues and cut-free sequent calculus derivations demonstrated in [15]. Recall also that the proofs in this article work with the de Bruijn syntax. The results in this section were mechanized in Coq at [17].

The majority of the soundness proof is straightforward. A slight difficulty arises from the differing treatment of quantifiers by material dialogues and sequent calculi. Compare a typical L$\forall$-rule with the state transition caused by the proponent attacking the admission $\dot\forall x.\varphi(x)$ with $A_{t^\rho}\,\varphi(x)$ and the opponent responding defending, both given below. While the sequent calculus simply instantiates the formula via a substitution $\varphi[t/x]$, material dialogues carry out the instantiation via the assignment ρ.

$$\mathrm{L}\forall\frac{\dot\forall x.\varphi \in \Gamma \quad \Gamma, \varphi[t/x] \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (\rho, \Gamma, C) \rightsquigarrow_{po} (\rho[x' \mapsto t^\rho], \varphi(x') :: \Gamma, C)$$

To prove soundness, one needs to show that these two methods of instantiation are "essentially the same". For this, we introduce *congruence relations* on different aspects of dialogues: Given assignments ρ, ρ' and formulas φ, φ', we define an equivalence relation $\rho, \varphi \equiv \rho', \varphi'$ which holds if φ and φ' are equal up to term evaluations in the respective assignments. This congruence can be extended to attacks and defenses and it can be shown that these relations do indeed "act as congruences" (see the mechanization for details). The following lemma is crucial to the proof of soundness.

Lemma 1. *Let (ρ, A, C) and (ρ', A', C') be dialogue states such that $\rho, A \equiv \rho', A'$ and $\rho, C \equiv \rho', C'$. If $Win(\rho, A, C)$ then $Win(\rho', A', C')$.*

Theorem 2 (Soundness). *Let Γ, φ be such that $\Gamma \Rightarrow \varphi$. Then $\Gamma \vDash^D \varphi$.*

Proof. For this, it suffices to show that for any standard structure $\mathbf{S}$

$$\begin{aligned}\Gamma \Rightarrow \Delta \to \forall \rho, A, C.\ (\forall \delta \in \Delta. \exists c \in C.\ c \rhd \delta \wedge (\forall \psi.\ \mathsf{adm}\, c = \ulcorner \psi \urcorner \to \psi \in A))\\ \to \Gamma \subseteq A \to \mathrm{Win}\,(\rho, A, C)\end{aligned}$$

because it allows one to conclude $\mathrm{Win}\,(\rho, c :: \Gamma, [c])$ for any $c \rhd \varphi$ — and thus $\mathrm{Win}\,(\rho, \Gamma, [], \varphi)$ — from $\Gamma \Rightarrow \varphi$. The proof proceeds per induction on $\Gamma \Rightarrow \Delta$. Lemma 1 is used in the cases of the quantifier rules. We encourage the curious reader to consult the mechanization [17].

We continue by proving completeness. We first prove that material validity entails validity on all classical, exploding models. This is extended to traditional completeness in Theorem 4 by use of completeness for classical exploding models on the $\forall, \to, \bot$-fragment (Theorem 1) and a de Morgan translation.

Lemma 2. *For any Γ and φ, $\Gamma \vDash^D \varphi$ entails $\Gamma \vDash^E \varphi$.*

Proof. Fix a classical, exploding structure $\mathbf{S}$. We extend Tarski satisfaction to defenses via $\rho \vDash D_A\, \varphi :\Leftrightarrow \rho \vDash \varphi$ and $\rho \vDash D_W\, \varphi\, s :\Leftrightarrow s \cdot \rho \vDash \varphi$ and $\rho \vDash D_M\, \varphi :\Leftrightarrow \rho \vDash \varphi$. Furthermore, we define an auxiliary predicate $\Gamma \vDash_\rho \bigvee \mathcal{D}$ on contexts Γ, assignments ρ and sets of defenses $\mathcal{D}$ which intuitively states that under the assignment ρ, Γ entails the disjunction of all semantic interpretations of $\mathcal{D}$:

$$\Gamma \vDash_\rho \bigvee \mathcal{D} \Leftrightarrow \rho \vDash \Gamma \to \forall \vec{s} : \mathbf{S}, \alpha.\ (\forall d \in \mathcal{D}.\ \rho \vDash d \to (\vec{s} \cdot \rho) \vDash \alpha) \to (\vec{s} \cdot \rho) \vDash \alpha$$

The proof relies on the two intermediate results below, whose proofs are rather routine. We refer the curious reader to the mechanization [17]. Here, $\hat{\mathbf{S}}$ denotes the standard structure with $\dot{\bot}^{\hat{\mathbf{S}}} = \bot$ which is otherwise exactly the same as $\mathbf{S}$.

(1) $\mathrm{Win}\,(\rho, \Gamma, C)$ in $\hat{\mathbf{S}}$ entails $\Gamma \vDash_\rho \bigvee(\bigcup_{c \in C} \mathcal{D}_c)$ in $\mathbf{S}$
(2) If $(c :: \Gamma) \vDash_\rho \bigvee(\mathcal{D}_c \cup \mathcal{D})$ for all $c \rhd \varphi$ then $\Gamma \vDash_\rho \bigvee(\{D_A\, \varphi\} \cup \mathcal{D})$ in $\mathbf{S}$

Now assume $\Gamma \vDash^D \varphi$ and a classical, exploding $\mathbf{S}, \rho$ with $\rho \vDash \Gamma$. Per assumption $\mathrm{Win}\,(\rho, \Gamma, [], \varphi)$ in $\hat{\mathbf{S}}$ meaning $(c :: \Gamma) \vDash_\rho \bigvee \mathcal{D}_c$ in $\mathbf{S}$ for all $c \rhd \varphi$ by (1) and thus $\Gamma \vDash_\rho \bigvee\{D_A \varphi\}$ by (2). By picking φ as α we then obtain $\rho \vDash \varphi$ in $\mathbf{S}$ as desired.

To conclude completeness, we employ a de Morgan translation from the full syntax of first-order logic into the $\dot\forall, \dot\to, \dot\bot$-fragment which is given below:

$$\bot^M := \bot \qquad (P\,\vec{s})^M := P\,\vec{s} \qquad (\varphi \land \psi)^M := \neg(\varphi^M \to \neg\psi^M)$$

$$(\varphi \lor \psi)^M := \neg\varphi^M \to \psi^M \qquad (\forall\varphi)^M := \forall\varphi^M \qquad (\exists\varphi)^M := \neg(\forall\neg\varphi^M)$$

$$(\varphi \to \psi)^M := \varphi^M \to \psi^M$$

Furthermore, a dialogical analogue of cut-admissibility is required to derive completeness on the full syntax. A formula φ *can be cut* if for any $\mathbf{S}, \rho, A, A', C$ with $\mathrm{Win}\,(\rho, A +\!\!+\, \varphi :: A', C)$ and $\mathrm{Win}\,(\rho, A +\!\!+\, A', C, \varphi)$ we also have that $\mathrm{Win}\,(\rho, A +\!\!+\, A', C)$. The proofs below rely on the congruence principles Lemma 1. As we believe that spelling out all applications of the principle obscures the simple ideas behind the proof, we opt to leave applications of Lemma 1 implicit where possible. Readers interested in the proofs in full detail may take a look at the Coq mechanization [17]. We prove dialogical cut-admissibility with the help of a lemma:

Lemma 3. *Let φ be such that all formulas of smaller complexity can be cut. Fix $c \rhd \varphi$ such that $\mathit{Win}\,(\rho, A, C +\!\!+\, c :: C')$ and for all $d \in \mathcal{D}_c$ justified under ρ we have $\mathit{Win}\,(d^O(\rho, A, C +\!\!+\, C'))$. Then $\mathit{Win}\,(\rho, A, C +\!\!+\, C')$.*

Proof. We prove a slight generalization: Instead of just $c \rhd \varphi$, we consider any n and $c \rhd (\uparrow^n\ \varphi)$ such that $\mathrm{Win}\,(\rho, A, C +\!\!+\, c :: C')$ and for all $d \in \mathcal{D}_c$ justified under ρ we have $\mathrm{Win}\,(d^O(\rho, A, C +\!\!+\, C'))$. We then proceed per induction on $\mathrm{Win}\,(\rho, A, C +\!\!+\, c :: C')$ and begin by performing a case distinction on the proponent's move in $\mathrm{Win}\,(\rho, A, C +\!\!+\, c :: C')$.

PA: The proponent uses $a \rhd \psi$ on $\psi \in A$. Then the proponent of $\mathrm{Win}\,(\rho, A, C +\!\!+ C')$ copies that move. There are two possible opponent responses.
  - In the case of $\mathsf{adm}\, a = \ulcorner\theta\urcorner$, the opponent may counter with some $c' \rhd \theta$. Then the proponent copies the strategy obtained from the inductive hypothesis upon the same counter.
  - The opponent may defend with some $d \in \mathcal{D}_a$. Then the proponent copies the strategy obtained for the inductive hypothesis upon the same defense.

PD: Then there is a $c' \in C +\!\!+ c :: C'$ and the proponent defends with some $d \in \mathcal{D}_c$. There are two cases to distinguish:
  $c' \in C +\!\!+ C'$: Then the proponent of $\mathrm{Win}\,(\rho, A, C +\!\!+ C')$ copies the defense. If d is not $D_M\,\varphi$ for some φ, then the opponent attacks the formula ψ admitted by d with some $a \rhd \psi$. The proponent then plays according to the strategy obtained from the inductive hypothesis upon $a \rhd \psi$.
  $c' = c$: Per assumption we have $\mathrm{Win}\,(d^O\,(\rho, A, C +\!\!+ C'))$. Then we perform a case distinction on the form of d.
    $d = D_M\,\varphi$: Then $\mathrm{Win}\,(d^O(\rho, A, C +\!\!+ C')) = \mathrm{Win}\,(\rho, A, C +\!\!+ C')$ and we are done.
    $d = D_A\,\psi$: The assumption thus is $\mathrm{Win}\,(\rho, \psi :: A, C +\!\!+ C')$. From the inductive hypothesis we obtain $\mathrm{Win}\,(\rho, A, C +\!\!+ C', \psi)$. As $D_A\,\psi \in \mathcal{D}_c$ and $c \rhd (\uparrow^n \varphi)$ we know that ψ is of lower complexity than φ, meaning it can be cut and we thus obtain $\mathrm{Win}\,(\rho, A, C +\!\!+ C')$.
    $d = D_W\,\psi\,s$: This case is analogous to that for $d = D_A\,\psi$ with a few more applications of Lemma 1.

Theorem 3 (Cut-admissibility). *All formulas can be cut.*

Proof. The proof proceeds per induction on formula complexity. Thus pick a φ such that all formulas of lower complexity can be cut. We show that

$$\mathrm{Win}\,(\rho, A +\!\!+ \uparrow^n \varphi :: A', C) \to \mathrm{Win}\,(\rho, A +\!\!+ A', C, \uparrow^n \varphi) \to \mathrm{Win}\,(\rho, A +\!\!+ A', C)$$

per induction on $\mathrm{Win}\,(\rho, A +\!\!+ \uparrow^n \varphi :: A', C)$ which subsumes the fact that φ can be cut. We perform a case distinction on the proponent move.

PA a Then the proponent attacks some $\psi \in A +\!\!+ \uparrow^n \varphi :: A'$ with $a \rhd \psi$. We distinguish two cases.
  $\psi \in A +\!\!+ A'$: Then the proponent of $\mathrm{Win}\,(\rho, A +\!\!+ A', C)$ copies that attack and proceeds per inductive hypothesis.
  $\psi = \uparrow^n \varphi$: Then $\mathrm{Win}\,(\rho, A +\!\!+ A', C, \uparrow^n \varphi)$ yields $\mathrm{Win}\,(\rho, A +\!\!+ A', a :: C)$ and the inductive hypothesis means that for all $d \in \mathcal{D}_a$ we have that $\mathrm{Win}\,(d^O\,(\rho, A +\!\!+ A', C))$. We may thus apply Lemma 3 to deduce $\mathrm{Win}\,(\rho, A +\!\!+ A', C)$.

PD ψ : Then there is some $c \in C$ and some $d \in \mathcal{D}_c$ such that d results in admitting ψ. The proponent of $\mathrm{Win}\,(\rho, A +\!\!+ A', C)$ thus copies that admission and proceeds per inductive hypothesis.

PM φ : Then $c \in C$ for the unique $a \rhd \varphi$ (in either case of $\varphi = \dot{\bot}$ or $\varphi = P\,\vec{t}$) and $\rho \vDash \varphi$ holds. The proponent of $\mathrm{Win}\,(\rho, A +\!\!+ A', C)$ can thus win as well by demonstrating $\rho \vDash \varphi$.

Combining Theorem 3 with the de Morgan translation above yields completeness on the full syntax of first-order logic.

Theorem 4 (Completeness). *For any Γ and φ, $\Gamma \vDash^D \varphi$ entails $\Gamma \Rightarrow \varphi$.*

Proof. First of all, $\Gamma \vDash^D \varphi$ entails $\Gamma^M \vDash^D \varphi^M$. This follows from the provability of $\varphi \dot{\leftrightarrow} \varphi^M$, soundness (Theorem 2) and cut-admissibility for material dialogues (Theorem 3). We can now apply Lemma 2 to obtain $\Gamma^M \vDash^E \varphi^M$, which entails $\Gamma^M \Rightarrow \varphi^M$ by Theorem 1. It is well known that this entails $\Gamma \Rightarrow \varphi$.

Observe that Theorem 4 was obtained fully constructively. This is noteworthy because similar results often make use of unconstructive principles. For example, the only method of extending completeness from the $\dot\forall, \dot\rightarrow, \dot\bot$-fragment to the full syntax for classical, standard Tarski models we know of uses LEM [5]. This should be taken as an indication that material dialogues are exceptionally well-suited as a semantics for classical first-order logic in a constructive setting.

4 Intuitionistic Material Dialogues

One of the striking features of dialogue games is that classical dialogue semantics can often be transformed into intuitionistic dialogue games by a simple change in the rules governing the interactions between proponent and opponent [10, 9]: The proponent may only ever defend against the opponent's most recent attack. This is analogous to the restriction to at most one right-hand formula for obtaining intuitionistic sequent calculi. The adjusted version of the proponent move transition relation $\rightsquigarrow_p$ for the arising intuitionistic material dialogues is given below. Note that in this section, in a slight abuse of notation, $\mathrm{Win}\,(\rho, A, C)$ and $\Gamma \vDash^D \varphi$ refer to dialogues played according to these intuitionistic rules. Further note that the rule PA remains unchanged when compared to classical material dialogues.

$$\text{PA}\,\frac{\varphi \in A \quad a \rhd \varphi}{(\rho, A, C) \rightsquigarrow_p (\rho, A, C)\,;PA\,a}$$

$$\text{PD}\,\frac{d \in \mathcal{D}_c \quad \rho \text{ justifies } d}{(\rho, A, c :: C) \rightsquigarrow_p d^P\,(\rho, A, c :: C)\,;\mathsf{move}\,d}$$

This version of intuitionistic material dialogues does not admit a constructive completeness proof. To demonstrate this, we define the following fragment of first-order logic:

$$a, b : \mathfrak{A} ::= \bot \mid P\,\vec{t} \mid a \dot\wedge b \mid a \dot\vee b \mid \dot\exists x.a \qquad\qquad P : \Sigma, \vec{t} : \mathfrak{T}^{|P|}$$
$$\varphi, \psi : \mathfrak{F}^{\mathsf{D}} ::= a \mid \varphi \dot\wedge \psi \mid \varphi \dot\vee \psi \mid a \dot\rightarrow \psi \mid \dot\forall x.\varphi \mid \dot\exists x.\varphi$$

$\mathfrak{A}$ is the fragment of first-order logic which allows attacking "blindly", i.e. the same attack pattern can be used on these formulas in every winning strategy. The fragment $\mathfrak{A}$ does not include $a \dot\rightarrow b$ as attacking it requires being able to defend a and $\dot\forall x.a$ as attacking it requires a (finite) choice of $s : \mathbf{S}$. Unless specified otherwise, we are working in a fixed standard structure $\mathbf{S}$.

Lemma 4. *For any $a : \mathfrak{A}$ and any ρ, A, C with $a \in A$ one may assume $\rho \vDash a$ to deduce $Win(\rho, A, C)$, i.e. $(\rho \vDash a \to Win(\rho, A, C)) \to Win(\rho, A, C)$.*

Proof. Per induction on the structure of a. Suppose $\rho \vDash a$ entails $\mathrm{Win}(\rho, a :: A, C)$ we show $\mathrm{Win}(\rho, a :: A, C)$. Where appropriate, we implicitly make use of the fact that $\mathrm{Win}(\rho, A, C)$ entails $\mathrm{Win}(\vec{s} \cdot \rho, A', \uparrow^n C)$ where $\vec{s} : \mathbf{S}$, $|\vec{s}| = n$ and $\uparrow^n A \subseteq A'$.

$a = P\,\vec{t}$: Then the proponent attacks $P\,\vec{t}$, forcing the opponent to demonstrate $\rho \vDash P\,\vec{t}$. The proponent may then continue according to the assumption.

$a = \dot{\bot}$: The proponent attacks $\dot{\bot}$ and wins.

$a = a \mathbin{\dot{\wedge}} b$: The proponent starts by attacking $a \mathbin{\dot{\wedge}} b$ with A_L and A_R, leaving us to prove that $\mathrm{Win}(\rho, a :: b :: A, C)$. Applying the IH for a and b means we may assume $\rho \vDash a$ and $\rho \vDash b$ to prove $\mathrm{Win}(\rho, a :: b :: A, C)$. As we thus know $\rho \vDash a \mathbin{\dot{\wedge}} b$ the proponent can proceed per assumption.

$a = a \mathbin{\dot{\vee}} b$: The proponent attacks $a \mathbin{\dot{\vee}} b$, leaving $\mathrm{Win}(\rho, c :: a \vee c :: A, C)$ for $c \in \{a, b\}$. Applying the IH for c allows us to assume $\rho \vDash c$, meaning the proponent can continue per assumption in either case.

$a = \dot{\exists}\, a$: The proponent attacks $\dot{\exists}\, a$, leaving $\mathrm{Win}(s \cdot \rho, a :: \uparrow(\dot{\exists}\, a :: A), \uparrow C)$. Per IH on a we may assume $s \cdot \rho \vDash a$ and thus $\rho \vDash \dot{\exists}\, a$, continuing per assumption.

Theorem 5. *Pick $\varphi : \mathfrak{F}^{\mathsf{D}}$, then $\rho \vDash \varphi$ entails $Win(\rho, A, C, \varphi)$ for any A and C.*

Proof. Proof per induction on φ.

$\varphi : \mathfrak{A}$: We only handle $\varphi = P\vec{t}$ and $\varphi = \dot{\bot}$ as the other cases are subsumed by other cases of this proof. If $\rho \vDash \dot{\bot}$ we are done. If $\varphi = P\vec{t}$ then the only possible challenge is $A_P\,\vec{t}$ to which the proponent responds by demonstrating $\rho \vDash P\vec{t}$.

$\varphi = \varphi \mathbin{\dot{\wedge}} \psi$: Then we know $\rho \vDash \varphi$ and $\rho \vDash \psi$. The possible challenges are A_L and A_R, defending against which leaves $\mathrm{Win}(\rho, A, A_X :: C, \theta)$ for some $\theta \in \{\varphi, \psi\}$. Either case holds per IH for θ.

$\varphi = \varphi \mathbin{\dot{\vee}} \psi$: Then we know $\rho \vDash \theta$ for $\theta \in \{\varphi, \psi\}$. The proponent thus defends against $A_\vee$ by admitting θ and proceeds per IH for θ.

$\varphi = a \mathbin{\dot{\to}} \psi$: Then we know $\rho \vDash a$ entails $\rho \vDash \psi$. In attacking, the opponent will admit a, leaving $\mathrm{Win}(\rho, a :: A, A_\to\, a\, \psi :: C)$. We apply Lemma 4, allowing us to assume $\rho \vDash a$ to prove $\mathrm{Win}(\rho, a :: A, A_\to\, a\, \psi :: C)$. The proponent thus defends by admitting ψ and proceeds per IH on ψ as $\rho \vDash \psi$ per assumption.

$\varphi = \dot{\forall}\, \varphi$: We know that $s \cdot \rho \vDash \varphi$ for any $s : \mathbf{S}$. The challenge will be $A_s\, \varphi$ for some $s : \mathbf{S}$. The proponent reacts by admitting $\varphi(s)$, proceeding per IH.

$\varphi = \dot{\exists}\, \varphi$: Then $s \cdot \rho \vDash \varphi$ for some $s : \mathbf{S}$. The only possible challenge is $A_\exists\, \varphi$ to which the proponent responds by admitting φ with s as the witness, proceeding per IH.

Theorem 5 can be made sense of in the following way: It is known that CIC is consistent with various non-intuitionistic intermediate logics whose axiom schemes lie partially in $\mathfrak{F}^{\mathsf{D}}$, e.g. classical logic $(a \mathbin{\dot{\vee}} \dot{\neg}\, a)$ and Gödel-Dummett logic LC $((a \mathbin{\dot{\to}} b) \mathbin{\dot{\vee}} (b \mathbin{\dot{\to}} a))$ for $a, b : \mathfrak{A}$. By Theorem 5, it is thus consistent with CIC that these parts of the axiom schemes are dialogically valid according to the rules investigated in this section. However, this means that completeness of these dialogues wrt. to some intuitionistic deduction system cannot be proven: If it could, that would mean it was consistent that parts of these non-intuitionistic axiom schemata, e.g. $P \dot{\vee} \dot{\neg} P$, were provable intuitionistically, which we know not to be the case.

Under the full law of the excluded middle, one can obtain an even stronger result: intuitionistic and classical dialogical validity, as defined here, fully coincide. This result relies on the following lemma:

Lemma 5. *Assuming LEM, the following holds for any φ in any standard* $\boldsymbol{S}$

1. $\forall \rho, A, C.\ \varphi \in A \to \rho \vDash \dot{\neg}\varphi \to \mathit{Win}(\rho, A, C)$
2. $\forall \rho, A, C.\ \rho \vDash \varphi \to \mathit{Win}(\rho, A, C, \varphi)$

Proof. We show prove both claims per simultaneous induction on φ. For most cases, 2. is the same as in Theorem 5 in which case we omit it. We write IHi for the inductive hypothesis for part i of Lemma 5.

$\varphi = P\,\vec{t}$: 1. The proponent may force the opponent to demonstrate $\rho \vDash P\,\vec{t}$ by attacking $P\,\vec{t} \in A$, contradicting $\rho \vDash \neg P\,\vec{t}$.

$\varphi = \dot{\bot}$: 1. The proponent may win by attacking $\dot{\bot} \in A$.

$\varphi = \varphi \dot{\to} \psi$:

1. Suppose $\rho \vDash \dot{\neg}(\varphi \dot{\to} \psi)$, meaning $\rho \vDash \varphi$ and $\rho \vDash \dot{\neg}\psi$. The proponent then attacks $\varphi \dot{\to} \psi \in A$. If the opponent counters the attack, the proponent can win by playing the strategy obtained by IH2 on $\rho \vDash \varphi$. If the opponent admits ψ, then the proponent plays according to IH1 on $\rho \vDash \dot{\neg}\psi$.
2. Suppose $\rho \vDash \varphi \dot{\to} \psi$. The opponent attacks $\varphi \dot{\to} \psi$ with $A_{\to}\,\varphi\,\psi$, admitting φ. By the law of the excluded middle, either $\rho \vDash \varphi$ or $\rho \vDash \dot{\neg}\varphi$. In the latter case, the proponent can now proceed per IH2 on $\rho \vDash \dot{\neg}\varphi$. In the former case we have $\rho \vDash \psi$ per assumption and the proponent can proceed by admitting ψ and playing along IH1 on $\rho \vDash \psi$.

$\varphi = \varphi \dot{\wedge} \psi$: 1. Suppose $\rho \vDash \dot{\neg}(\varphi \dot{\wedge} \psi)$, meaning $\rho \vDash \dot{\neg}\varphi$ or $\rho \vDash \dot{\neg}\psi$. The proponent attacks the side of the contradicted formula of $\varphi \dot{\wedge} \psi \in A$ and proceeds per IH1.

$\varphi = \varphi \dot{\vee} \psi$: 1. Suppose $\rho \vDash \dot{\neg}(\varphi \dot{\vee} \psi)$, meaning $\rho \vDash \dot{\neg}\varphi$ and $\rho \vDash \dot{\neg}\psi$. By attacking $\varphi \dot{\vee} \psi \in A$, the proponent thus forces the opponent to admit either clause, being able to proceed via IH1 in either case.

$\varphi = \dot{\forall}\varphi$: If $\rho \vDash \dot{\neg}\dot{\forall}\varphi$ that means there is an $s : \mathbf{S}$ with $s \cdot \rho \vDash \dot{\neg}\varphi$. The proponent thus attack $\dot{\forall}\varphi$ with $A_s\,\varphi$ and proceeds per IH1.

$\varphi = \dot{\exists}\varphi$: Suppose $\rho \vDash \dot{\neg}\dot{\exists}\varphi$, meaning $s \cdot \rho \vDash \dot{\neg}\varphi$ for any $s : \mathbf{S}$. Then the proponent attacks $\dot{\exists}\varphi \in A$ and proceeds per IH1.

Theorem 6. *Under LEM, classical and intuitionistic dialogical validity agree.*

Proof.

$\leftarrow$: This is the case — even without the law of excluded middle — as every intuitionistic winning strategy is also a classical winning strategy on the same state.

→: Suppose $\Gamma \vDash^D \varphi$ classically. By Lemma 2 this means $\Gamma \vDash^E \varphi$. As every standard structure is exploding and under the LEM every structure is classical, this means φ is valid under Γ in every standard structure. By Lemma 5 this entails that $\Gamma \vDash^D \varphi$ intuitionistically.

Note that the failure of completeness is not simply due to the "wrong choice of rules" for intuitionistic dialogues. While there are examples of the proponent restriction failing to turn classical dialogues intuitionistic (see e.g. [18]) we do not believe this to be the case in this instance. Formal dialogues for intuitionistic first-order logic, which are obtained from their classical counterparts via that very restriction, are constructively sound and complete [3, 4]. Rather, we believe the cause of failure lies in the difference between formal and material dialogues: their treatment of atomic formulas.

5 Discussion

Mechanization of active research While researching for this article, we mechanized the results from Section 3 in the interactive theorem prover Coq. The mechanization can be found at [17]. Mechanizing the results of Section 3 revealed some mistakes in our initial definition of the rules for material dialogues which, albeit being minor, invalidated both soundness and completeness. We missed these mistakes while working "on paper" and believe it would have taken much longer to discover them without the mechanization. Having machine checked the definitions in Section 3 gave us sufficient confidence in the correctness of the technical details of material dialogues to work solely on paper for the remainder of the article. It should also be noted that the mechanization took up only about a quarter of the overall time spent researching, in large part due to building on top of the a large preexisting mechanization from [5]. We believe this might be a worthwhile trade-off between the time requirement of a full mechanization of all results and the room for error in working solely on paper.

Proof strategies for Completeness We prove completeness by relating dialogical validity to validity in a model-theoretic semantics and appealing to a preexisting completeness result. This is the quickest way to obtain completeness in the framework set up by [5]. For classical material dialogues, we believe it would also be possible to obtain a direct constructive completeness proof with regards to natural deduction on the basis of a Henkin construction.

Benefits of Material Dialogues When working with Tarski semantics in CIC, one's attention needs to be restricted to classical structures (as define in Section 2). Many structures of interest such as the standard model of Peano arithmetic is not provably classical in constructive settings and can thus not be studied in a Tarski setting. In contrast, Section 3 demonstrates that classical material dialogues embody classical logic regardless of the classicality of the underlying structure. It thus seems like a promising basis on which to carry out model-theoretic investigations of classical first-order logic in constructive settings. However, we have not yet investigated these possibilities more deeply.

Faithfulness to Lorenzen's material dialogues We attempted to be as faithful to Lorenzen's definitions from [12, 13] as possible while implementing material dialogues played over first-order structures. Arguably, this second aspect already is in conflict with Lorenzen's ideas as he placed a lot of value in the "underlying game" for settling atomic propositions to be of a discrete nature, something completely lost in our formulation. However, all the attacks and defenses for the connectives of first-order logic are exactly as they are in Lorenzen's work. Notably, our usage of a structure defined, standard constant $\dot{\bot}^{\mathbf{S}}$ is very similar to Lorenzen's definitions which propose to fix some unwinnable game as a stand-in for a "demonstration" of $\dot{\bot}$.

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