Constructive Analysis of Material Dialogues

Dominik Wehr, University of Amsterdam
supervised by Dominik Kirst, Saarland University

September 11, 2020

Dialogues, first introduced by Paul Lorenzen [1,2], characterize logical validity in terms of debates about formulas. If the debate associated with a formula φ can always be "won", φ is considered valid. Dialogues take place between two participants, taking turns either admitting a formula or challenging one of their opponent's admissions. In Lorenzen's original presentation of dialogues, later dubbed "material dialogues", participants reduced their opponent's claims to claims of atomic formulas, which would then be settled in a previously agreed upon model. This presentation has since fallen out of fashion in favor of the purely syntactic "formal dialogues" introduced by Lorenzen's student Kuno Lorenz [3]. Albeit dialogues were originally intended to serve as a semantics for intuitionistic logic, it was soon discovered that a slight modifications of the rules of engagement gave rise to a semantics for classical logic as well.

In this project, we give formal renderings of material dialogues for first-order logic in the state-transition-system style we have employed in prior work with formal dialogues [4]. We prove their soundness and completeness to demonstrate their suitability as semantics of first-order logic. As we work within the Calculus of Inductive Constructions as captured by the interactive theorem prover Coq, special attention is paid to the constructivity of the completeness proofs. In prior work, we have shown that some semantics lend themselves to fully constructive completeness proofs, while others require various non-constructive axioms [4, 5]. We thus aim to reveal where material dialogues fall within that spectrum. Although we initially conjectured that material dialogue will not admit constructive completeness proofs, our analysis reveals the opposite to be the case.

1. Overview

We begin by quickly summarizing the results of the project. Note that we employ non-standard terminology, speaking of proving a semantics sound or complete when showing that some deduction system is sound or complete with regards to it. We do this to stress that we are analyzing the semantics, not the deduction system.

Classical material dialogues In Section 3, we formalize classical material dialogues for first-order logic. In Section 3.1, we prove them sound with regards to a cut-free, classical sequent calculus. Interestingly, classical material dialogues are sound on any first-order structure, whereas classical Tarski semantics require the underlying structure to satisfy the axioms of classical predicate logic (a property not necessarily held by all structures in a constructive setting). One could thus say that the "classicality" of classical material dialogues is within their rules of engagement, not the underlying structures. In Section 3.2 we prove that classical material validity entails exploding classical Tarski validity, a constructively stricter notion than standard classical Tarski validity. We then use the constructive completeness of exploding classical Tarski validity on the $\forall, \rightarrow, \bot$-fragment [6] to deduce the same for classical material dialogues. As a corollary, we deduce that standard classical Tarski validity entailing classical material validity is equivalent to the unconstructive Markov's principle. In Section 3.3 we extend the completeness result for classical material dialogues to the full syntax of first-order logic via the DeMorgan translation. For this, we prove a dialogical rendering of cut-elimination.

Intuitionistic material dialogues In Section 4 we analyze intuitionistic material dialogues as first put forward by Lorenzen. We prove that standard Tarski validity on the fragment F^D given below entails intuitionistic material validity.

$$a, b : \mathsf{A} ::= \bot \mid P\,\overline{t} \mid a \wedge b \mid a \vee b \mid \exists a \qquad\qquad P : \Sigma, \overline{t} : \mathsf{T}$$
$$\varphi, \psi : \mathsf{F}^\mathsf{D} ::= a \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid a \rightarrow \psi \mid \forall \varphi \mid \exists \varphi$$

This means that intuitionistic material dialogues "inherit" parts of the meta-logical setting. This culminates in the fact that under the law of the excluded middle, intuitionistic and classical material dialogues fully coincide. There thus cannot be a constructive proof of completeness without additional axioms forcing the CIC to behave truly intuitionistically, making intuitionistic material dialogues ill-suited as a semantics of intuitionistic first-order logic.

Kripke material dialogues In reaction to the results of Section 4, we propose an alternative dialogical semantics in Section 5. As classical material dialogues could be considered "classical dialogues played on Tarski structures", we consider intuitionistic dialogues played on Kripke structures. We demonstrate their suitability by deriving many of the same results for them as for the classical material dialogues of Section 3. In Section 5.1 we prove them sound with regards to a cut-free intuitionistic sequent calculus. In Section 5.2, we show that Kripke material validity entails exploding Kripke validity. We use the constructive completeness of exploding Kripke models for the $\forall, \rightarrow, \bot$-fragment [7] to deduce the same for Kripke material dialogues. Similarly, this means that standard Kripke validity entailing Kripke material validity is equivalent to the non-constructive Markov's principle.

2. Preliminaries

Pick some **signature** Σ of n-ary constants c and predicates P. Then we define an associated **term and formula language** with DeBruijn [8] binders.

$$t : \mathsf{T} ::= n \mid c\,\bar{t} \qquad\qquad n : \mathbf{N}, c : \Sigma$$

$$\varphi : \mathsf{F} ::= \bot \mid P\,\bar{t} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \forall \varphi \mid \exists \varphi \qquad\qquad P : \Sigma$$

A **structure** $\mathbf{S}$ consists of a type X, a predicate interpretation $P^{\mathbf{S}} : X \rightarrow \ldots \rightarrow X \rightarrow \mathbf{P}$ for each $P \in \Sigma$, a constant interpretation $c^{\mathbf{S}} : X \rightarrow \ldots \rightarrow X \rightarrow X$ for each $c \in \Sigma$ and an absurdity interpretation $\bot^{\mathbf{S}}$. A **model** $\mathbf{M}$ consists of a structure $\mathbf{S}$ together with an assignment $\rho : \mathbf{N} \rightarrow X$.

We define the usual term evaluation function t^{ρ} inside a structure $\mathbf{S}$

$$x^{\rho} := \rho\, x \qquad (f\,\bar{t})^{\rho} := f^{\mathbf{S}}\,\bar{t}^{\rho}$$

and the Tarski satisfation relation $\rho \vDash \varphi$

$$\rho \vDash P\,\bar{t} \Leftrightarrow P^{\mathbf{S}}\,\bar{t}^{\rho} \qquad \rho \vDash \varphi \rightarrow \psi \Leftrightarrow \rho \vDash \varphi \rightarrow \rho \vDash \psi$$

$$\rho \vDash \varphi \wedge \psi \Leftrightarrow \rho \vDash \varphi \wedge \rho \vDash \psi \qquad \rho \vDash \varphi \vee \psi \Leftrightarrow \rho \vDash \varphi \vee \rho \vDash \psi$$

$$\rho \vDash \forall \varphi \Leftrightarrow \forall s : \mathbf{S},\ (s \cdot \rho) \vDash \varphi \qquad \rho \vDash \exists \varphi \Leftrightarrow \exists s : \mathbf{S},\ (s \cdot \rho) \vDash \varphi$$

$$\rho \vDash \bot \Leftrightarrow \bot^{\mathbf{S}}$$

We call a structure $\mathbf{S}$ **classical** if for all environments ρ and formulas φ, ψ it can be shown that $\rho \vDash ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$. We call a structure $\mathbf{S}$ **exploding** if for all environments ρ and formulas φ it can be shown that $\rho \vDash \bot \rightarrow \varphi$. We call a structure $\mathbf{S}$ **standard** if $\bot^{\mathbf{S}}$ is contradictory. Note that all standard structures are exploding. We call a formula φ **valid in classical exploding models** in a context Γ, written $\Gamma \vDash^{E} \varphi$, if for all classical, exploding structures $\mathbf{S}$ and all environments ρ we have that $\rho \vDash \Gamma$ entails $\rho \vDash \varphi$. Validity in classical standard models, $\Gamma \vDash^{S} \varphi$, is defined analogously.

A **Kripke frame** consists of a type K and a preorder $\leq : K \rightarrow K \rightarrow \mathbf{P}$ on that type. A **Kripke structure** consists of a Kripke frame $(K, \leq)$ and a functor $\iota : K \rightarrow \mathcal{S}$ from it into the category of structures. For a $k : K$ we write $\mathbf{S}_k$ for $\iota\, k$ and for a $s : \mathbf{S}_k$ we write $s^{k'}$ for $\iota\,(k \leq k')\, s$ where applicable. A **Kripke model** consists of a Kripke structure, a world k and an environment in $\mathbf{S}_k$. Given a Kripke model, we define a relation $\rho^{k} \Vdash \varphi$, denoting that φ is forced at a world k under the k-environment ρ as below

$$\rho^k \Vdash P\,\bar{t} \Leftrightarrow P^k\,\bar{t}^\rho \qquad \rho^k \Vdash \varphi \to \psi \Leftrightarrow \forall k \le k'.\ \rho^{k'} \Vdash \varphi \to \rho^{k'} \Vdash \psi$$
$$\rho^k \Vdash \varphi \wedge \psi \Leftrightarrow \rho^k \Vdash \varphi \wedge \rho^k \Vdash \psi \qquad \rho^k \Vdash \varphi \vee \psi \Leftrightarrow \rho^k \Vdash \varphi \vee \rho^k \Vdash \psi$$
$$\rho^k \Vdash \exists\varphi \Leftrightarrow \exists s : \mathbf{S}_k,\ (s \cdot \rho^k) \vDash \varphi \qquad \rho^k \Vdash \forall\varphi \Leftrightarrow \forall k \le k', s : \mathbf{S}_{k'},\ (s \cdot \rho^{k'}) \Vdash \varphi$$
$$\rho^k \Vdash \bot \Leftrightarrow \bot^k$$

We call a Kripke structure exploding or standard if all images of ι are such. We call a formula φ **valid in exploding models** in a context Γ, written $\Gamma \vDash^K \varphi$, if for all exploding Kripke structures, all worlds $k : K$ and all k-environments ρ we have that $\rho^k \Vdash \Gamma$ entails $\rho^k \Vdash \varphi$. We denote validity in standard models with $\Gamma \vDash^{KS} \varphi$.

3. Classical Material Dialogues

We begin by giving a formal rendering of classical material dialogues. A material dialogue always takes place in an ambient model $\mathbf{S}, \rho$. We write $a \rhd \varphi$ to mean that a is an **attack** on φ. Some attacks force the attacker to also **admit** a formula. This is formalized through a function $\operatorname{adm} : \mathcal{A} \to \mathcal{O}(\mathsf{F})$ where $\operatorname{adm} a = \ulcorner\varphi\urcorner$ means that φ needs to be admitted in the process of attacking with a and $\operatorname{adm} a = \emptyset$ means no admission needs to be made. Each attack a has an associated set $\mathcal{D}_a$ called its **defenses**. There are three different kinds of defenses: $D_A\,\varphi$ denotes simply admitting the formula φ, $D_W\,\varphi\,s$ denotes admitting $\varphi(s)$ where $s : \mathbf{S}$. Lastly, $D_M\,P\,\bar{t}$ means claiming to be able to demonstrate that $P\,\bar{t}$ holds in the ambient model.

$$\mathbf{D} ::= D_A\,\varphi \mid D_W\,\varphi\,s \mid D_M\,P\,\bar{t} \qquad \varphi : \mathsf{F}, s : \mathbf{S}, P : \Sigma, \bar{t} : \mathsf{T}$$

Below are the attacks and defenses for first-order logic. We define $\operatorname{adm}(A_\to\,\varphi\,\psi) = \ulcorner\varphi\urcorner$ and $\operatorname{adm} a = \emptyset$ for all other attacks.

$$A_\bot \rhd \bot \qquad \mathcal{D}_{A_\bot} = \{\} \qquad A_\to\,\varphi\,\psi \rhd \varphi \to \psi \qquad \mathcal{D}_{A_\to\,\varphi\,\psi} = \{D_A\,\psi\}$$
$$A_L\,\varphi \rhd \varphi \wedge \psi \qquad \mathcal{D}_{A_L\,\varphi} = \{D_A\,\varphi\} \qquad A_\vee\,\varphi\,\psi \rhd \varphi \vee \psi \qquad \mathcal{D}_{A_\vee\,\varphi\,\psi} = \{D_A\,\varphi, D_A\,\psi\}$$
$$A_R\,\psi \rhd \varphi \wedge \psi \qquad \mathcal{D}_{A_R\,\psi} = \{D_A\,\psi\} \qquad A_s\,\varphi \rhd \forall\varphi \qquad \mathcal{D}_{A_s\,\varphi} = \{D_W\,\varphi\,s\}$$
$$A_\exists\,\varphi \rhd \exists\varphi \qquad \mathcal{D}_{A_\exists\,\varphi} = \{D_W\,\varphi\,s \mid s : \mathbf{S}\} \qquad A_P\,\bar{t} \rhd P\,\bar{t} \qquad \mathcal{D}_{A_P\,\bar{t}} = \{D_M\,P\,\bar{t}\}$$

We formalize classical material dialogues as a turn taking game between two players. The **proponent** tries to defend the validity of some formula, whereas the **opponent** tries to challenge the proponent's claims in such a way that they can't respond. All dialogues we consider are **E-dialogues** which restrict the opponent to only ever react to the proponent's previous move. It can be shown that the notion of validity given by E-dialogues is equivalent to that of the more intuitive D-dialogues, in which this restriction is lifted [5,9].

We model the dialogue game as a state transition system. Pick some structure **S**. We call a triple $(\rho, A, C) : (\mathbf{N} \to \mathbf{S}) \times \mathcal{L}(\mathsf{F}) \times \mathcal{L}(\mathcal{A})$ a **state**. Together $\mathbf{S}, \rho$ form the **ambient model**. The list A contains all of the opponent's admissions while C records all attacks that the opponent has leveled against the proponent.

Each round, the proponent gets to make a move, defending against a challenge previously issued by the opponent, either by admitting a formula (PD) or by demonstrating that an atomic formula holds in the ambient model (PM), or challenging one of the opponent's admissions (PA). We define a defense's effect on the game state as a function d^P as below:

$$(D_A\, \varphi)^P\, s = s \qquad (D_W\, \varphi\, s)\, (\rho, A, C) = (s \cdot \rho, \Uparrow\! A, \Uparrow\! C) \qquad (D_M\, P\, \bar{t})\, s = s$$

Here, we use the shifting operation on attacks, defined as follows:

$$\Uparrow\! A_\bot = A_\bot \qquad \Uparrow\!(A_\to\, \psi) = A_\to \Uparrow\!\psi \qquad \Uparrow\!(A_L\, \varphi) = A_L \Uparrow\!\varphi \qquad \Uparrow\!(A_R\, \psi) = A_R \Uparrow\!\psi$$

$$\Uparrow\!(A_\vee\, \varphi\, \psi) = A_\vee \Uparrow\!\varphi \Uparrow\!\psi \qquad \Uparrow\!(A_s\, \varphi) = A_s(\varphi[0 \cdot \Uparrow\! \mathrm{id}]) \qquad \Uparrow\!(A_\exists\, \varphi) = A_\exists(\varphi[0 \cdot \Uparrow\! \mathrm{id}])$$

Similarly we define a function mapping each defense to a proponent move

$$\mathrm{move}\, (D_A\, \varphi) = PD\, \varphi \qquad \mathrm{move}\, (D_W\, \varphi\, s) = PD\, \varphi \qquad \mathrm{move}\, (D_M\, P\, \bar{t}) = PM\, P\, \bar{t}$$

Lastly, we say ρ **justifies** d is d is not of the shape $D_M\, P\, \bar{t}$ or if $P^{\mathbf{S}}\, \bar{t}^{\rho}$ holds. These definitions allow us to give a simple definition of the state transitions a proponent can trigger by making a move.

$$\text{PA}\,\frac{\varphi \in A \quad a \rhd \varphi}{(\rho, A, C) \rightsquigarrow_p (\rho, A, C)\, ; PA\, a} \qquad \text{PD}\,\frac{c \in C \quad d \in \mathcal{D}_c \quad \rho \text{ justifies } d}{(\rho, A, C) \rightsquigarrow_p d^P\, (\rho, A, C)\, ; \mathrm{move}\, d}$$

The opponent must react to the proponent's move. If the proponent defended by admitting a formula, they must issue a new challenge against that formula (OA). If the proponent attacked one of their admissions, they can either defend against that attack (OD) or counter the attack, meaning attacking the admission made by the opponent in issuing the attack (OC). If the proponent demonstrated the validity of an atomic formula in the ambient model, the opponent cannot respond at all. We define an operation d^O analogously to that for the proponent and use it to define the transition steps the opponent can trigger by making a move. In a slight abuse of notation, we write $c :: A$, where c is an attack, for $\psi :: A$ if $\mathrm{adm}\, c = \ulcorner \psi \urcorner$ and A if $\mathrm{adm}\, c = \emptyset$.

$$(D_A\, \varphi)\, (\rho, A, C) = (\rho, \varphi :: A, C) \qquad (D_W\, \varphi\, s)\, (\rho, A, C) = (s \cdot, \varphi :: \Uparrow\! A, \Uparrow\! C) \qquad (D_M\, P\, \bar{t})\, s = s$$

$$\text{OA}\,\frac{c \rhd \varphi}{(\rho, A, C)\, ; PD\, \varphi \rightsquigarrow_o (\rho, \psi :: A, c :: C)} \qquad \text{OC}\,\frac{a \rhd \varphi \quad \mathrm{adm}\, a = \ulcorner \psi \urcorner \quad \psi \rhd c}{(\rho, A, C)\, ; PA\, a \rightsquigarrow_o (\rho, c :: A, c :: C)}$$

$$\text{OD}\,\frac{d \in \mathcal{D}_a \quad \rho \text{ justifies } d}{(\rho, A, C)\, ; PA\, a \rightsquigarrow_o d^O\, (\rho, A, C)}$$

A state can be **won** if the proponent can make a move such that all possible states resulting from an opponent response to that move can be won. We can define this as an inductive predicate which is very similar in flavor to the usual definition of well-foundedness of a relation.

$$\frac{s \rightsquigarrow_p s' \,;\, m \quad \forall s''.\, s' \,;\, m \rightsquigarrow_o s'' \longrightarrow \mathrm{Win}\, s''}{\mathrm{Win}\, s}$$

We extend the notion to **winning formulas** φ with Win (ρ, A, C, φ) meaning that for all attacks $c \triangleright \varphi$ we have Win $(\rho, c :: A, c :: C)$. A formula φ is **valid** in a context Γ, written $\Gamma \models^D \varphi$, if for all structures $\mathbf{S}$ and environments ρ we have Win $(\rho, A, [], \varphi)$.

3.1. Soundness

We prove that classical material dialogues are sound with regards to the the cut-free classical sequent calculus LK (given below). Note that this is the easiest soundness result to obtain as winning strategies of dialogues always carry the "structure" of a cut-free sequent calculus, as elegantly demonstrated by [10]. Proving soundness with regards to a system with cuts, say a natural deduction system, would thus necessitate giving a proof of dialogical cut-elimination first.

$$\mathrm{Ax}\,\frac{P\,\bar{t} \in \Gamma \quad P\,\bar{t} \in \Delta}{\Gamma \Rightarrow \Delta} \qquad \mathrm{L}\bot\,\frac{\bot \in \Gamma}{\Gamma \Rightarrow \Delta} \qquad \mathrm{L}\!\to\,\frac{\varphi \to \psi \in \Gamma \quad \Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\mathrm{R}\!\to\,\frac{\varphi \to \psi \in \Delta \quad \Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta} \qquad \mathrm{L}\wedge\,\frac{\varphi \wedge \psi \in \Gamma \quad \Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\mathrm{R}\wedge\,\frac{\varphi \wedge \psi \in \Delta \quad \Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \Delta} \qquad \mathrm{L}\vee\,\frac{\varphi \vee \psi \in \Gamma \quad \Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\mathrm{R}\vee\,\frac{\varphi \vee \psi \in \Delta \quad \Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \Delta} \qquad \mathrm{L}\forall\,\frac{\forall\varphi \in \Gamma \quad \Gamma, \varphi[t] \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\mathrm{R}\forall\,\frac{\forall\varphi \in \Delta \quad \Uparrow\Gamma \Rightarrow \varphi, \Uparrow\Delta}{\Gamma \Rightarrow \Delta} \qquad \mathrm{L}\exists\,\frac{\exists\varphi \in \Gamma \quad \Uparrow\Gamma, \varphi \Rightarrow \Uparrow\Delta}{\Gamma \Rightarrow \Delta}$$

$$\mathrm{R}\exists\,\frac{\exists\varphi \in \Delta \quad \Gamma \Rightarrow \varphi[t], \Delta}{\Gamma \Rightarrow \Delta}$$

Note the difference between the L∀ rule and the state resulting in the proponent attacking an admitted universally quantified formula, which correspond to each other in the soundness proof. The premise of the L∀ rule, presented as a dialogue state is $(\rho, \varphi[t] :: A, C)$ whereas the state resulting from the opponent reacting to an attack on an admission $\forall\varphi$ is $(t^\rho \cdot \rho, \varphi :: \Uparrow A, \Uparrow C)$. To prove soundness, we need to show that these two states are "essentially the same". For this, we introduce various **congruence relations** on different parts of dialogues. Given environments ρ, ρ' and formulas φ, φ', we define an equivalence relation $\rho, \varphi \equiv_f \rho', \varphi'$ as below. Intuitively, $\rho, \varphi \equiv_f \rho', \varphi'$ means that φ and φ' are equal up to their terms and the pairs of terms at a certain position within these formulas are equal under evaluation in the respective environment.

$$\frac{}{\rho, \bot \equiv_f \rho', \bot} \qquad \frac{\bar{t}^{\rho} = \bar{t'}^{\rho'}}{\rho, P\,\bar{t} \equiv_f \rho', P\,\bar{t'}} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi' \quad \rho, \psi \equiv_f \rho', \psi'}{\rho, \varphi \Box \psi \equiv_f \rho', \varphi' \Box \psi'}$$

$$\frac{\forall d.\ (d \cdot \rho), \varphi \equiv_f (d \cdot \rho'), \varphi'}{\rho, \Box\varphi \equiv_f \rho', \Box\varphi'}$$

We then extend this congruence to attacks $\rho, a \equiv_a \rho', a'$, defenses $\rho, d \equiv_d \rho', d'$ and show that these relations do indeed "act as congruences" (see Appendix A for details). These congruences give rise to a lemma that is crucial for the proof of soundness.

Lemma 1 Let (ρ, A, C) and (ρ', A', C') be dialogue states such that $\rho, A \equiv_f \rho', A'$ and $\rho, C \equiv_a \rho', C'$. If $\mathrm{Win}\,(\rho, A, C)$ then $\mathrm{Win}\,(\rho', A', C')$ as well.

Theorem 2 (Soundness) Let Γ, φ be such that $\Gamma \Rightarrow \varphi$. Then $\Gamma \models^D \varphi$.

Proof For this, it suffices to show that for any structure $\mathbf{S}$ we have

$$\Gamma \Rightarrow \Delta \to \forall \rho, A, C.\ (\forall \delta \in \Delta. \exists c \in C.\ c \rhd \delta \land (\forall \psi.\ \mathrm{adm}\, c = \ulcorner \psi \urcorner \to \psi \in A))$$
$$\to \Gamma \subseteq A \to \mathrm{Win}\,(\rho, A, C)$$

As this means that $\Gamma \Rightarrow \varphi$ entails $\mathrm{Win}\,(\rho, c :: \Gamma, [c])$ for any $c \rhd \varphi$ and thus $\mathrm{Win}\,(\rho, \Gamma, [], \varphi)$.

We prove the claim per induction on $\Gamma \Rightarrow \Delta$ and only spell out an exemplary subset of the cases. For a full proof, the reader may consult the accompanying formalization.

Ax: Then $P\,\bar{t} \in \Gamma$ and $P\,\bar{t} \in \Delta$. The proponent thus first attacks the admission $P\,\bar{t}$, forcing the opponent to demonstrate $P^{\mathbf{S}}\,\bar{t}^{\rho}$. With this, the proponent can now justify the defense against the challenge against $P\,\bar{t} \in \Delta$, leaving the opponent without any way of responding and thus winning the dialogue.

L$\bot$: Then $\bot \in \Gamma$. Then the proponent can attack $\bot$, leaving the opponent without a way of defending against the attack and thus winning the dialogue.

L$\to$: Then $\varphi \to \psi \in \Gamma$ and we obtain inductive hypotheses for $\Gamma \Rightarrow \varphi, \Delta$ and $\Gamma, \psi \Rightarrow \Delta$. The proponent thus attacks the admission $\varphi \to \psi$. The opponent has two ways of responding to this attack:

- If the opponent defends against the attack by admitting ψ then the proponent can win by playing the strategy obtained from the inductive hypothesis on $\Gamma, \psi \Rightarrow \Delta$.
- If the opponent counters, attacking the admission φ with some challenge $c \rhd \varphi$ then the proponent can win by playing the strategy obtained from the inductive hypothesis on $\Gamma \Rightarrow \varphi, \Delta$.

R$\rightarrow$: Then $\varphi \rightarrow \psi \in \Delta$ and we have an IH for $\Gamma, \varphi \Rightarrow \psi, \Delta$. The proponent thus defends against the challenge on $\varphi \rightarrow \psi$ to which the opponent must respond with a challenge $c \rhd \psi$. The proponent can thus win by playing according to the IH as φ has already been admitted per our assumption.

L$\forall$: Then $\forall\varphi \in \Gamma$ and we have an IH for $\Gamma, \varphi[t] \Rightarrow \Delta$ for some term t. The proponent thus attacks $\forall\varphi$ with $A_{t^\rho}\, \varphi$ against which the opponent must defend with $D_W\, \varphi\, t^\rho$. The state resulting from this is $(t^\rho \cdot \rho, \varphi :: \uparrow A, \uparrow C)$. However, the IH only yields a winning strategy for the state $(\rho, \varphi[t] :: A, C)$. We can now apply Lemma 1 as $t^\rho \cdot \rho, \varphi :: \uparrow A \equiv_f \rho, \varphi[t] :: A$ and $t^\rho \cdot \rho, \uparrow C \equiv_a \rho, C$ to transform the winning strategy provided by the IH into the desired form.

R$\forall$: Then $\forall\varphi \in \Delta$ and we have an IH for $\uparrow\Gamma \Rightarrow \varphi, \uparrow\Delta$. The proponent thus defends with $D_W\, \varphi\, s$ for some $s : \mathbf{S}$ and the opponent responds with some challenge $c \rhd \varphi$. This results in the state $(s \cdot \rho, c :: \uparrow A, c :: \uparrow C)$ for which the IH directly provides a winning strategy. ∎

3.2. Completeness

We prove completeness for classical material dialogues. We first show that validity in classical dialogues entails Tarski validity in classical exploding models. We then use prior work of Herbelin and Ilik [6] to obtain completeness with regards to the $\forall, \rightarrow, \bot$-fragment of first-order logic. For this, we extend the Tarski satisfaction relation to defenses.

$$\rho \models D_A\, \varphi \Leftrightarrow \rho \models \varphi \qquad \rho \models D_W\, \varphi\, s \Leftrightarrow (s \cdot \rho) \models \varphi \qquad \rho \models D_M\, P\, \overline{t} \Leftrightarrow \rho \models P\, \overline{t}$$

We define an auxiliary predicate on contexts Γ, environments ρ and sets of defenses Δ

$$\Gamma \models_\rho \bigvee \Delta \Leftrightarrow \rho \models \Gamma \rightarrow \forall \overline{s} : \mathbf{S}, \alpha.\ (\forall d \in \Delta.\ \rho \models d \rightarrow (\overline{s} \cdot \rho) \models \alpha) \rightarrow (\overline{s} \cdot \rho) \models \alpha$$

As the notation suggests, $\Gamma \models_\rho \bigvee \Delta$ is an impredicative formalization of the claim that under the environment ρ, Γ entails the disjunction of all semantical interpretations of Δ. That the proof of Theorem 4 below will fail for the much simpler definition of $\exists d \in \Delta.\ \rho \models d$ in the cases of the proponent attacking $\varphi \vee \psi$ and $\exists\varphi$. We first show a useful lemma before moving on to the full proof.

Lemma 3 Pick some classical, exploding structure $\mathbf{S}$, environment ρ, context Γ, set Δ and formula φ. If for all $c \rhd \varphi$ we have $(c :: \Gamma) \models_\rho \bigvee(\mathcal{D}_c \cup \Delta)$ then $\Gamma \models_\rho \bigvee(\{D_A\, \varphi\} \cup \Delta)$.

Proof We assume (1) $\forall c \rhd \varphi.\ (c :: \Gamma) \models_\rho \bigvee(\mathcal{D}_c \cup \Delta)$. To show $\Gamma \models_\rho \bigvee\{D_A\, \varphi\} \cup \Delta$ we assume (2) $\forall d \in \{D_A\, \varphi\} \cup \Delta.\ \rho \models d \rightarrow (\overline{s} \cdot \rho) \models \alpha$ and $\rho \models \Gamma$ to deduce $(\overline{s} \cdot \rho) \models \alpha$. The proof proceeds per case distinction on φ.

$\varphi = \bot$: For $A_\bot$ assumption (1) yields $\Gamma \models_\rho \bigvee \Delta$ which together with (2) proves the claim.

$\varphi = P\,\bar{t}$: This follows as (2) and (1) with $A_P\,\bar{t}$ are equivalent as $\rho \models D_M\,P\,\bar{t} \Leftrightarrow \rho \models P\,\bar{t}$.

$\varphi = \varphi \to \psi$: As **S** is classical, we may apply Peirce's law and assume $(\bar{s} \cdot \rho) \models \neg\alpha$. Per (2), it suffices to show $\rho \models \varphi \to \psi$. We thus assume $\rho \models \varphi$ and apply (1) for $A_{\to}\,\varphi\,\psi$, yielding $(\varphi :: \Gamma) \models_\rho \bigvee\{D_A\,\psi\} \cup \Delta$, to deduce $\rho \models \psi$. For the case $d = D_A\,\psi$ this is trivial, thus suppose $d \in \Delta$. By (2), $\rho \models d$ entails $(\bar{s} \cdot \rho) \models \alpha$ and thus $\bot^{\mathbf{S}}$, meaning $\rho \models \psi$ as **S** is exploding.

$\varphi = \varphi \wedge \psi$: With (1) for $A_L\,\varphi$ and $A_R\,\psi$ together with (2) we can deduce $(\bar{s} \cdot \rho) \models \alpha \vee \uparrow^n \varphi$ and $(\bar{s} \cdot \rho) \models \alpha \vee \uparrow^n \psi$ where $n = |\bar{s}|$. If $(\bar{s} \cdot \rho) \models \alpha$ we are done. If $(\bar{s} \cdot \rho) \models \uparrow^n \varphi$ and $(\bar{s} \cdot \rho) \models \uparrow^n \psi$ this means $\rho \models \varphi \wedge \psi$ and thus $(\bar{s} \cdot \rho) \models \alpha$ by (2).

$\varphi = \varphi \vee \psi$: Applying (1) for $A_\vee\,\varphi\,\psi$ together with (2) directly yields the claim.

$\varphi = \forall\varphi$: This case proceeds analogous to that of $\varphi \to \psi$: Suppose $(\bar{s} \cdot \rho) \models \neg\alpha$ via Peirce's law, apply (2), leaving $s' \cdot \rho \models \varphi$ to be proven and apply (1) with $A_{s'}\,\varphi$ to achieve this.

$\varphi = \exists\varphi$: This case proceeds analogous to that of $\varphi \wedge \psi$: Apply (1) with $A_\exists\,\varphi$ to deduce $(\bar{s} \cdot \rho) \models \alpha \vee \uparrow^n (\exists\varphi)$ and close using (2). ■

Note that the ability to extend the context in $\Gamma \models_\rho \Delta$ is used in the $\forall\varphi$ case of the proof above. Indeed, this is the only where we make use of it overall.

Theorem 4 Let **S** be classical and exploding and (ρ, A, C) a state. If $\mathrm{Win}\,(\rho, A, C)$ then $A \models_\rho \bigvee \mathcal{D}_C$ where $\mathcal{D}_C = \bigcup_{c \in C} \mathcal{D}_c$.

Proof We proceed per induction on $\mathrm{Win}\,(\rho, A, C)$ and perform a case distinction on the proponent move. We assume (H) $\forall d \in \mathcal{D}_C.\ \rho \models d \to (\bar{s} \cdot \rho) \models \alpha$ and $\rho \models A$. We only handle some of the cases.

$PA\,a$: Then there is some $\varphi \in A$ with $a \rhd \varphi$. We perform a case distinction on φ.

- $\varphi = \bot$: Then $\bot \in A$ and thus $\rho \models \bot$.
- $\varphi = P\,\bar{t}$: Then $\rho \models P\,\bar{t}$ and per IH $(P\,\bar{t} :: A) \models_\rho \bigvee \mathcal{D}_C$. We may apply this, together with (H) to deduce $(\bar{s} \cdot \rho) \models \alpha$.
- $\varphi = \varphi \to \psi$: Then the IH upon the opponent countering together with Lemma 3 yields (1) $A \models_\rho \{D_A\,\varphi\} \cup \mathcal{D}_C$ and upon the opponent defending is (2) $(\psi :: A) \models_\rho \mathcal{D}_C$. We first assume $(\bar{s} \cdot \rho) \models \neg\alpha$ by Peirce's law and then apply (2). This leaves us proving $\rho \models \psi$ which we can do by proving $\rho \models \varphi$ as $\varphi \to \psi \in A$. We conclude this from (1), (H) and $(\bar{s} \cdot \rho) \models \neg\alpha$.
- $\varphi = \forall\varphi$: The IH yields $(\varphi :: \uparrow A) \models_{s' \cdot \rho} \mathcal{D}_{\uparrow C}$ for some $s' : \mathbf{S}$. We apply this to deduce $(\bar{s} \cdot \rho) \models \alpha$ using the fact that $\rho \models \forall\varphi$ and thus $s' \cdot \rho \models (\varphi :: \uparrow A)$. (H) can be adapted to $\uparrow C$ as $\rho \models \psi \Leftrightarrow s' \cdot \rho \models \uparrow\psi$ for all formulas ψ.

PD φ : Then defending results in a state (ρ', A', C') and the IH together with Lemma 3 yields $A' \models_{\rho'} \{D_A\,\varphi\} \cup \mathcal{D}_{C'}$. We further know that there is a $c \in C$ and a defense $d \in \mathcal{D}_c$ such that $\rho \models d \Leftrightarrow \rho' \models \varphi$. We may thus apply the IH to resolve the claim as $D_A\,\varphi$ does not "add anything" to $\mathcal{D}_{C'}$ and the validity of (H) is maintained under the possible transformations applied to (ρ, A, C).

PM : Then $P^{\mathbf{S}}\bar{t}^{\rho}$ holds and there is a $c \in C$ with $c \rhd P\,\bar{t}$. Then we may apply (H) with $P_M\,P\,\bar{t} \in \mathcal{D}_C$ to deduce $(\bar{s} \cdot \rho) \models \alpha$. ∎

Corollary 5 For any Γ and φ, $\Gamma \models^D \varphi$ entails $\Gamma \models^E \varphi$.

Proof Assume $\Gamma \models^D \varphi$ and pick some classical, exploding model $\mathbf{S}, \rho$ and suppose $\rho \models \Gamma$. Per assumption Win $(\rho, \Gamma, [], \varphi)$ meaning $(c :: \Gamma) \models_\rho \bigvee \mathcal{D}_c$ for all $c \rhd \varphi$ by Theorem 4 and thus $\Gamma \models_\rho \bigvee\{D_A\varphi\}$ by Lemma 3. By picking φ as α we then obtain $\rho \models \varphi$. ∎

We proceed by deriving a proper completeness result by relying on prior work by Herbelin and Ilik [6]. We take $\Gamma \vdash \varphi$ to be a classical natural deduction system.

Corollary 6 (Fragment Completeness) When restricting to the $\forall, \rightarrow, \bot$-fragment, $\Gamma \models^D \varphi$ entails $\Gamma \vdash \varphi$.

Proof For this, first notice that the proof of Corollary 5 remains correct when the syntax is restricted to the $\forall, \rightarrow, \bot$-fragment, meaning $\Gamma \models^E \varphi$. Herbelin and Ilik have shown in [6] that this implies $\Gamma \vdash \varphi$. ∎

We close this section by analyzing the relationship between Tarski validity in *standard* classical models and classical dialogical validity. It is easy to see that dialogical validity is subsumed by standard Tarski validity.

Corollary 7 Whenever $\Gamma \models^D \varphi$ then also $\Gamma \models^S \varphi$.

Proof This follows from Corollary 5 by noting that every standard model is exploding. ∎

We now prove the converse direction to be unconstructive by proving it equivalent to completeness of $\Gamma \models^S \varphi$ which we have in turn proven unconstructive in [4]. We call a theorem **unconstructive** if it can be shown to be equivalent to a proof principle independent of the CIC. In this case, the equivalent unconstructive principle is the **Markov's principle**, the principle of double negation elimination restricted to the halting problem for a Church-Turing notion of computation, such as the the untyped lambda calculus (more details can be found in [4]).

Lemma 8 $\Gamma \models^S \varphi$ entails $\Gamma \models^D$ if and only if $\Gamma \models^S \varphi$ entails $\Gamma \vdash \varphi$.

Proof $\rightarrow$: This follows directly from Corollary 13.

$\leftarrow$: From $\Gamma \models^S \varphi$ we know $\Gamma \vdash \varphi$ which entails $\Gamma \Rightarrow \varphi$ via cut elimination and $\Gamma \models^D \varphi$ by Theorem 2. ∎

3.3. Dialogical cuts

We conclude our exploration of classical material dialogues by extending the completeness result to the full syntax of first-order logic. For this, we prove the dialogical equivalent of cut-elimination. We say a formula φ **can be cut** if for any ρ, A, A', C we have that $\text{Win}\,(\rho, A +\!\!+ \varphi :: A', C)$ and $\text{Win}\,(\rho, A +\!\!+ A', C, \varphi)$ entails $\text{Win}\,(\rho, A +\!\!+ A', C)$. We prove full cut elimination in two steps.

The proofs in this section heavily rely on the weakening principles Lemmas 1 and 9. However, we feel that spelling out all applications of these principles obscures the simple ideas behind this section's proofs. We thus opt to leave applications of Lemmas 1 and 9 implicit where possible. Readers interested in the proofs in full detail may take a look at the Coq mechanization accompanying this report.

Lemma 9 (Weakening) Let $\text{Win}\,(\rho, A, C)$ and $A \subseteq A'$, $C \subseteq C'$ then $\text{Win}\,(\rho, A', C')$.

Proof Simple induction on $\text{Win}\,(\rho, A, C)$. ∎

Lemma 10 Pick a formula φ such that all formulas of smaller complexity can be cut. Now pick some n and $c \rhd (\uparrow^n \varphi)$ such that $\text{Win}\,(\rho, A, C +\!\!+ c :: C')$ and for all $d \in \mathcal{D}_c$ justified under ρ we have $\text{Win}\,(d^O(\rho, A, C +\!\!+ C'))$. Then $\text{Win}\,(\rho, A, C +\!\!+ C')$.

Proof We proceed per induction on $\text{Win}\,(\rho, A, C +\!\!+ c :: C')$. We first perform a case distinction on the proponent's move in $\text{Win}\,(\rho, A, C +\!\!+ c :: C')$.

PA: The proponent uses $a \rhd \psi$ on some $\psi \in A$. Then the proponent of $\text{Win}\,(\rho, A, C +\!\!+ C')$ copies that move. There are two possible opponent responses.

- In the case of adm $a = \ulcorner \theta \urcorner$, the opponent may counter with some $c' \rhd \theta c'$. Then the proponent copies the strategy obtained from the inductive hypothesis upon the same counter.
- The opponent may defend with some $d \in \mathcal{D}_a$. Then the proponent copies the strategy obtained for the inductive hypothesis upon the same defense.

PD: Then there is a $c' \in C +\!\!+ c :: C'$ and the proponent defends with some $d \in \mathcal{D}_c$. There are two cases to distinguish:

$c' \in C +\!\!+ C'$: Then the proponent of $\text{Win}\,(\rho, A, C +\!\!+ C')$ copies the defense. If d is not $D_M\,P\,\bar{t}$ for some $P, \bar{t}$, then the opponent attacks the formula ψ admitted by d with some $a \rhd \psi$. The proponent then plays according to the strategy obtained from the inductive hypothesis upon $a \rhd \psi$.

$c' = c$: Per assumption we have $\text{Win}\,(d^O\,(\rho, A, C +\!\!+ C'))$. Then we perform a case distinction on the form of d.

$d = D_M\,P\,\bar{t}$: Then $\text{Win}\,(d^O(\rho, A, C +\!\!+ C')) = \text{Win}\,(\rho, A, C +\!\!+ C')$ and we are done.

$d = D_A\,\psi$: The assumption thus is $\text{Win}\,(\rho, \psi :: A, C +\!\!+ C')$. From the inductive hypothesis we obtain $\text{Win}\,(\rho, A, C +\!\!+ C', \psi)$. As $D_A\,\psi \in \mathcal{D}_c$ and $c \rhd (\uparrow^n \varphi)$ we know that ψ is of lower complexity than φ, meaning it can be cut and we thus obtain $\text{Win}\,(\rho, A, C +\!\!+ C')$.

$d = D_W\,\psi\,s$: This case is analogous to that for $d = D_A\,\psi$ with a few more applications of Lemma 1. ■

Theorem 11 (Dialogical Cut) All formulas can be cut.

Proof The proof proceeds per induction on formula complexity. Thus pick a φ such that all formulas of lower complexity can be cut. We show that

$$\text{Win}\,(\rho, A +\!\!+ \uparrow^n \varphi :: A', C) \to \text{Win}\,(\rho, A +\!\!+ A', C, \uparrow^n \varphi) \to \text{Win}\,(\rho, A +\!\!+ A', C)$$

per induction on $\text{Win}\,(\rho, A +\!\!+ \uparrow^n \varphi :: A', C)$ which subsumes the fact that φ can be cut. We perform a case distinction on the proponent move.

PA a Then the proponent attacks some $\psi \in A +\!\!+ \uparrow^n \varphi :: A'$ with $a \rhd \psi$. We distinguish two cases.

$\psi \in A +\!\!+ A'$: Then the proponent of $\text{Win}\,(\rho, A +\!\!+ A', C)$ copies that attack and proceeds per inductive hypothesis.

$\psi = \uparrow^n \varphi$: Then $\text{Win}\,(\rho, A +\!\!+ A', C, \uparrow^n \varphi)$ yields $\text{Win}\,(\rho, A +\!\!+ A', a :: C)$ and the inductive hypothesis means that for all $d \in \mathcal{D}_a$ we have that $\text{Win}\,(d^O\,(\rho, A +\!\!+ A', C))$. We may thus apply Lemma 10 to deduce $\text{Win}\,(\rho, A +\!\!+ A', C)$.

PD ψ : Then there is some $c \in C$ and some $d \in \mathcal{D}_c$ such that d results in admitting ψ. The proponent of $\text{Win}\,(\rho, A +\!\!+ A', C)$ thus copies that admission and proceeds per inductive hypothesis.

PM $P\,\bar{t}$: Then $A_P\,\bar{t} \in C$ and $P^{\mathbf{S}}\,\bar{t}^{\rho}$ holds. The proponent of $\text{Win}\,(\rho, A +\!\!+ A', C)$ can thus win as well by demonstrating $P^{\mathbf{S}}\,\bar{t}^{\rho}$. ■

Note that this cut-elimination principle would also allow us to prove soundness with regards to classical natural deduction without much effort.

To extend the completeness result to the all connectives, we employ a DeMorgan translation, similar to our approach to the same problem for Tarski semantics in [5]. In contrast to that approach, the translation process will be fully constructive. We define the DeMorgan translation of a formula φ^D as follows:

$$\bot^D := \bot \qquad (P\,\bar{s})^D := P\,\bar{s} \qquad (\varphi \to \psi)^D := \varphi^D \to \psi^D \qquad (\varphi \land \psi)^D := \neg(\varphi^D \to \neg\psi^D)$$

$$(\varphi \lor \psi)^D := \neg\varphi^D \to \psi^D \qquad (\forall\varphi)^D := \forall\varphi^D \qquad (\exists\varphi)^D := \neg(\forall\neg\varphi^D)$$

Lemma 12 If $\text{Win}(\rho, A, C, \varphi)$ then $\text{Win}(\rho, A^D, C, \varphi^D)$.

Proof We proceed in two steps. We first show that $\text{Win}(\rho, A^D, C, \varphi)$ and from this that $\text{Win}(\rho, A^D, C, \varphi^D)$.

1. We prove the generalization $\text{Win}(\rho, A + B, C, \varphi) \to \text{Win}(\rho, A + B^D, C, \varphi)$ per induction on B. The case of $B = []$ is trivial, thus suppose $B = \psi :: B'$. By Lemma 9 we know that $\text{Win}(\rho, A + \psi :: \psi^D :: B', C, \varphi)$. It is well known that $\psi^D \Rightarrow \psi$ and by Theorem 2 and Lemma 9 thus $\text{Win}(\rho, A + \psi^D :: B', C, \psi)$. We may thus apply Theorem 11 to cut ψ and obtain $\text{Win}(\rho, A + \psi^D :: B', C, \varphi)$. We then continue per inductive hypothesis with the choice $A = A + [\psi^D]$.
2. It is well known that $\varphi \Rightarrow \varphi^D$ thus $\text{Win}(\rho, \varphi :: A^D, C, \varphi^D)$ and by Theorem 2 and Lemma 9. We may now apply Theorem 11 with $\text{Win}(\rho, A^D, C, \varphi)$ to cut φ and obtain $\text{Win}(\rho, A^D, C, \varphi^D)$ ∎

Corollary 13 (Completeness) For any Γ and φ, $\Gamma \models^D \varphi$ entails $\Gamma \vdash \varphi$.

Proof By Lemma 12, $\Gamma \models^D \varphi$ entails $\Gamma^D \models^D \varphi^D$. We may now apply Corollary 6 to obtain $\Gamma^D \vdash \varphi^D$ which can easily be shown to entail $\Gamma \vdash \varphi$ (for example, we do this in [5]). ∎

4. Intuitionistic Material Dialogues

Intuitionistic material dialogues differ from their classical counterparts by the restriction that the proponent may only defend against the opponent's most recent attack. This is the dialogical analogue to the restriction to at most one left formula in the intuitionistic sequent calculus. The proponent's possible moves are thus as given below.

$$\text{PA}\;\frac{\varphi \in A \quad a \rhd \varphi}{(\rho, A, C) \rightsquigarrow_p (\rho, A, C)\,;\, PA\, a} \qquad \text{PD}\;\frac{d \in \mathcal{D}_c \quad \rho \text{ justifies } d}{(\rho, A, c :: C) \rightsquigarrow_p d^P\,(\rho, A, c :: C)\,;\, \text{move}\, d}$$

Intuitionistic material dialogues do not admit a constructive completeness proof. Indeed, in a classical setting they are incomplete with regards to intuitionistic first-order logic. To demonstrate this, we define a fragment of first-order logic as follows:

$$a, b : \mathsf{A} ::= \bot \mid P\,\bar{t} \mid a \wedge b \mid a \vee b \mid \exists a \qquad P : \Sigma, \bar{t} : \mathsf{T}$$
$$\varphi, \psi : \mathsf{F}^{\mathsf{D}} ::= a \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid a \to \psi \mid \forall \varphi \mid \exists \varphi$$

Unless specified otherwise, we are working in a standard structure **S**.

Lemma 14 $\text{Win}(\rho, A, C)$ entails $\text{Win}(\bar{s} \cdot \rho, A', \uparrow^n C)$ where $\bar{s} : \mathbf{S}$, $|\bar{s}| = n$ and $\uparrow^n A \subseteq A'$.

Proof Per induction on $\text{Win}(\rho, A, C)$. ∎

Lemma 15 For any $a : \mathsf{A}$ and any ρ, A, C with $a \in A$ one may assume $\rho \models a$ to deduce $\text{Win}(\rho, A, C)$.

Proof Per induction on the structure of a. Suppose $\rho \models a$ entails $\text{Win}(\rho, a :: A, C)$ we show $\text{Win}(\rho, a :: A, C)$. We implicitly apply Lemma 14 where appropriate.

$a = P\,\bar{t}$: Then the proponent attacks $P\,\bar{t}$, forcing the opponent to admit $\rho \models P\,\bar{t}$. The proponent may then continue according to the assumption.

$a = \bot$: The proponent attacks $\bot$ and wins.

$a = a \wedge b$: The proponent starts by attacking $a \wedge b$ with A_L and A_R, leaving us to prove that $\text{Win}(\rho, a :: b :: A, C)$. Applying the IH for a and b means we may assume $\rho \models a$ and $\rho \models b$ to prove $\text{Win}(\rho, a :: b :: A, C)$. As we thus know $\rho \models a \wedge b$ the proponent can proceed per assumption.

$a = a \vee b$: The proponent attacks $a \vee b$, leaving $\text{Win}(\rho, c :: a \vee c :: A, C)$ for $c \in a, b$. Applying the IH for c allows us to assume $\rho \models c$, meaning the proponent can continue per assumption in either case.

$a = \exists a$: The proponent attacks $\exists a$, leaving $\text{Win}(s \cdot \rho, a :: \uparrow(\exists a :: A), \uparrow C)$. Per IH on a we may assume $s \cdot \rho \models a$ and continue per assumption. ∎

Note that A is the fragment of F which allows attacking "blindly", meaning the same attack pattern can be used on these formulas in every winning strategy. This fragment does not include $a \longrightarrow b$ as attacking it requires being able to defend a and $\forall a$ as attacking it requires a (finite) choice of $s : \mathbf{S}$.

Theorem 16 Pick $\varphi : \mathsf{F}^{\mathsf{D}}$, then $\rho \models \varphi$ entails $\text{Win}(\rho, A, C, \varphi)$ for any A and C.

Proof Proof per induction on φ.

$\varphi = a$: We only handle $\varphi = P\bar{t}$ and $\varphi = \bot$ as the other cases are subsumed by other cases of this proof. If $\rho \models \bot$ we are done. If $\rho \models P\bar{t}$ then the only possible challenge is $A_P\,\bar{t}$ to which the proponent can respond by admitting $\rho \models P\bar{t}$.

$\varphi = \varphi \wedge \psi$: Then we know $\rho \models \varphi$ and $\rho \models \psi$. The possible challenges are A_L and A_R, defending against which leaves $\text{Win}(\rho, A, A_X :: C, \theta)$ for some $\theta \in \{\varphi, \psi\}$. Either case holds per IH for θ.

$\varphi = \varphi \vee \psi$: Then we know $\rho \models \theta$ for $\theta \in \{\varphi, \psi\}$. The proponent thus defends against $A_\vee$ by admitting θ and proceeds per IH for θ.

$\varphi = a \longrightarrow \psi$: Then we know $\rho \models a$ entails $\rho \models \psi$. In attacking, the opponent will admit a, leaving $\text{Win}(\rho, a :: A, A_\to\, a\,\psi :: C)$. We apply Lemma 15, allowing us to assume $\rho \models a$ to prove $\text{Win}(\rho, a :: A, A_\to\, a\,\psi :: C)$. The proponent thus defends by admitting ψ and proceeds per IH on ψ as $\rho \models \psi$ per assumption.

$\varphi = \forall\varphi$: We know that $s \cdot \rho \models \varphi$ for any $s : \mathbf{S}$. The challenge will be $A_s\,\varphi$ for some $s : \mathbf{S}$. The proponent thus reacts by admitting φ, proceeding per IH.

$\varphi = \exists\varphi$: Then $s \cdot \rho \models \varphi$ for some $s : \mathbf{S}$. The only possible challenge is $A_\exists\,\varphi$ to which the proponent responds by admitting φ with s as the witness, proceeding per IH. ■

Note that F^D can be simplified, up to intuitionistic equivalence, by taking A to be only $P\,\bar{t}$ and $\bot$. We opted to demonstrate the result for the more complex fragment as it makes it more apparent why we chose exactly this fragment (the "blind attack" justification). Note also that we will not be able to extend the above result to all of F constructively as this would subsume the translation we have show to be unconstructive in Lemma 8.

Importantly, Theorem 16 means that if the meta-logic is at least as strong as some non-intuitionistic intermediate logics, their axiom schemata for formulas for $a, b : \mathsf{A}$ are valid in intuitionistic dialogues. We list some examples below.

- Classical logic C: $a \vee \neg a$
- Gödel-Dummett logic LC: $(a \to b) \vee (b \to a)$
- Logics of bounded cardinality BC_n: $\bigvee_{i=1}^{n} \bigwedge_{j<i} a_j \to a_i$
- Logics of bounded width BW_n: $\bigvee_{i=1}^{n} \bigwedge_{j \neq i} a_j \to a_i$
- Logics of bounded depth BD_n: $a_n \vee (a_n \to (a_{n-1} \vee (a_{n-1} \to ...(a_2 \vee (a_2 \to (a_1 \vee \neg a_1))))))$

As these intermediate logics are consistent with the CIC, there is no hope of proving the completeness of intuitionistic material dialogues with regards to some intuitionistic deduction system over the CIC without additional assumptions as this would contradict the aforementioned consistencies. However, there might be such a proof under axioms guaranteeing the CIC to behave truly intuitionistically.

We can obtain an even stronger result: Under full classical logic, intuitionistic and classical dialogical validity coincide.

Lemma 17 Under the law of the excluded middle, the following holds for any formula φ in any standard structure

1. $\forall \rho, A, C.\ \rho \models \neg\varphi \to \varphi \in A \to \mathrm{Win}\,(\rho, A, C)$
2. $\forall \rho, A, C.\ \rho \models \varphi \to \mathrm{Win}\,(\rho, A, C, \varphi)$

Proof We show both claims simultaneously per induction on φ. For most cases, 2. works the same as in Theorem 16 in which case we omit them.

$\varphi = P\,\bar{t}$: 1. The proponent may force the opponent to demonstrate $\rho \models P\,\bar{t}$ by attacking $P\,\bar{t} \in A$, contradicting $\rho \models \neg P\,\bar{t}$.

$\varphi = \bot$: 1. The proponent may win by attacking $\bot \in A$.

$\varphi = \varphi \to \psi$:

1. Suppose $\rho \models \neg(\varphi \to \psi)$, meaning $\rho \models \varphi$ and $\rho \models \neg\psi$. The proponent then attacks $\varphi \to \psi \in A$. If the opponent counters the attack, the proponent can win by playing the strategy obtained by IH2 on $\rho \models \varphi$. If the opponent admits ψ, then the proponent plays according to IH1 on $\rho \models \neg\psi$.
2. Suppose $\rho \models \varphi \to \psi$. The opponent attacks $\varphi \to \psi$ with $A_{\to}\, \varphi\, \psi$, admitting φ. By the law of the excluded middle, either $\rho \models \varphi$ or $\rho \models \neg\varphi$. In the latter case, the proponent can now proceed per IH2 on $\rho \models \neg\varphi$. In the former case we have $\rho \models \psi$ per assumption and the proponent can proceed by admitting ψ and playing along IH1 on $\rho \models \psi$.

$\varphi = \varphi \land \psi$: 1. Suppose $\rho \models \neg(\varphi \land \psi)$, meaning $\rho \models \neg\varphi$ or $\rho \models \neg\psi$. The proponent thus attacks the side of the contradicted formula of $\varphi \land \psi \in A$ and proceeds per IH1.

$\varphi = \varphi \lor \psi$: 1. Suppose $\rho \models \neg(\varphi \lor \psi)$, meaning $\rho \models \neg\varphi$ and $\rho \models \neg\psi$. By attacking $\varphi \lor \psi \in A$, the proponent thus forces the opponent to admit either clause, thus being able to proceed via IH1.

$\varphi = \forall\varphi$: If $\rho \models \neg\forall\varphi$ that means there is an $s : \mathbf{S}$ with $s \cdot \rho \models \neg\varphi$. The proponent thus attack $\forall\varphi$ with $A_s\, \varphi$ and proceeds per IH1.

$\varphi = \exists\varphi$: Suppose $\rho \models \neg\exists\varphi$, meaning $s \cdot \rho \models \neg\varphi$ for any $s : \mathbf{S}$. Then the proponent attacks $\exists\varphi \in A$ and proceeds per IH1. ■

Corollary 18 Under the law of the excluded middle, classical and intuitionistic dialogical validity coincide.

Proof $\leftarrow$: This is the case even without the law of excluded middle as every winning strategy for an intuitionistic material dialogue is a winning strategy for the classical material dialogues on the same state.

$\to$: Suppose $\Gamma \models^D \varphi$ classically. In Corollary 5 we have shown that this means $\Gamma \models^E \varphi$. As every standard structure is exploding and under the LEM every structure is classical, this means φ is valid under Γ in every standard structure. By Lemma 17 this means that $\Gamma \models^D \varphi$ intuitionistically. ■

5. Kripke Material Dialogues

In the previous section we demonstrate that intuitionistic material dialogues fail at capturing intuitionistic first-order logic. In this section, we give an alternative dialogical system which succeeds in this. Classical material dialogues can be seen as classical dialogues played on Tarski models, the canonical notion of model for classical first-order logic. In that vein, we present intuitionistic dialogues played on Kripke models as a semantics for intuitionistic first-order logic. These stray far from the ideas of Lorenzen but are none the less interesting in their own right.

A Kripke material dialogue is played on a Kripke structure $K, \leq, \iota$. The game states of Kripke material dialogues are dependent pairs $(k, \rho, A, C) : \Sigma k : K,\ \mathbf{N} \to \mathbf{S}_k \times \mathcal{L}(\mathsf{F}) \times \mathcal{L}(\mathcal{A})$ which can be viewed as material dialogue states "at a world k in K". To mirror the clauses of $\Vdash$ for $\to$ and $\forall$, opponent attacks in Kripke dialogues can move the game state along $\leq$ in K. To express this we define a predicate $a|k \mapsto k'$ with where $A_{\to}\ \varphi\ \psi|k \mapsto k'$ and $A_s\ \varphi|k \mapsto k'$ hold whenever $k \leq k'$ and $a|k \mapsto k$ holds for all other attacks a. The definitions of valid moves and their effects on the game state are largely analogous to intuitionistic material dialogues, this time also incorporating the movement along the Kripke frame. Note that the definition of d^P and d^O are essentially the same as for the previous material dialogues, leaving the world unchanged.

$$\text{PA}\ \frac{\varphi \in A \qquad a \rhd \varphi}{(k, \rho, A, C) \rightsquigarrow_p (k, \rho^k, A, C) \,;\, PA\ a}$$

$$\text{PD}\ \frac{d \in \mathcal{D}_c \qquad \rho \text{ justifies } d}{(k, \rho, A, c :: C) \rightsquigarrow_p d^P\ (k, \rho, A, c :: C) \,;\, \text{move } d}$$

$$\text{OA}\ \frac{c \rhd \varphi \qquad c|k \mapsto k'}{(k, \rho, A, C) \,;\, PD\ \varphi \rightsquigarrow_o (k', \rho^{k'}, \psi :: A, c :: C)}$$

$$\text{OC}\ \frac{a \rhd \varphi \qquad \text{adm}\ a = \ulcorner \psi \urcorner \qquad \psi \rhd c \qquad c|k \mapsto k'}{(k, \rho, A, C) \,;\, PA\ a \rightsquigarrow_o (k', \rho^{k'}, c :: A, c :: C)}$$

$$\text{OD}\ \frac{d \in \mathcal{D}_a \qquad \rho \text{ justifies } d}{(k, \rho, A, C) \,;\, PA\ a \rightsquigarrow_o d^O\ (k, \rho, A, C)}$$

The definition of $\text{Win}\ (k, \rho, A, C)$ essentially remains unchanged. We modify the definition of $\text{Win}\ (k, \rho, A, C, \varphi)$ to be that we have $\text{Win}\ (k', \rho, a :: A, c :: C)$ for any $a \rhd \varphi$ with $a|k \mapsto k'$. We then define $\Gamma \models^D \varphi$ the same as before.

5.1. Soundness

We first prove that Kripke material dialogues are sound with regards to the intuitionistic sequent calculus LJ given below. We again prefer using a sequent calculus as the deduction system for the soundness proof as this means we do not need to prove Kripke dialogical cut elimination first.

$$\textsc{Ax}\,\frac{P\,\bar{t} \in \Gamma}{\Gamma \Rightarrow P\,\bar{t}} \qquad \textsc{L}\bot\,\frac{\bot \in \Gamma}{\Gamma \Rightarrow \delta} \qquad \textsc{L}{\to}\,\frac{\varphi \to \psi \in \Gamma \quad \Gamma \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow \delta}{\Gamma \Rightarrow \delta}$$

$$\textsc{R}{\to}\,\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi} \qquad \textsc{L}\land\,\frac{\varphi \land \psi \in \Gamma \quad \Gamma, \varphi, \psi \Rightarrow \delta}{\Gamma \Rightarrow \delta} \qquad \textsc{R}\land\,\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \land \psi}$$

$$\textsc{L}\lor\,\frac{\varphi \lor \psi \in \Gamma \quad \Gamma, \varphi \Rightarrow \delta \quad \Gamma, \psi \Rightarrow \delta}{\Gamma \Rightarrow \delta} \qquad \textsc{R}\lor\textsc{L}\,\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \lor \psi} \qquad \textsc{R}\lor\textsc{R}\,\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \lor \psi}$$

$$\textsc{L}\forall\,\frac{\forall\varphi \in \Gamma \quad \Gamma, \varphi[t] \Rightarrow \delta}{\Gamma \Rightarrow \delta} \qquad \textsc{R}\forall\,\frac{\Uparrow\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \forall\varphi} \qquad \textsc{L}\exists\,\frac{\exists\varphi \in \Gamma \quad \Uparrow\Gamma, \varphi \Rightarrow \uparrow\delta}{\Gamma \Rightarrow \delta}$$

$$\textsc{R}\exists\,\frac{\Gamma \Rightarrow \varphi[t]}{\Gamma \Rightarrow \exists\varphi}$$

Note that this proof of soundness requires the congruence machinery we describe in Appendix A as captured by Lemma 19.

Lemma 19 Suppose $\text{Win}\,(k, \rho, A, C)$ and $\rho, A \equiv_f \rho', A'$, $\rho, C \equiv_a \rho', C'$ for some rho', A' and C'. Then $\text{Win}\,(k, \rho', A', C')$.

Theorem 20 Suppose $\Gamma \Rightarrow \delta$ then $\Gamma \models^D \delta$.

Proof For this, we prove $\Gamma \Rightarrow \delta \to \forall k, \rho, C.\ \text{Win}\,(k, \rho, \Gamma, \delta)$ per induction on $\Gamma \Rightarrow \delta$. We again only handle a few exemplary cases.

Ax: The only possible challenge is $A_P\,\bar{t}$. Then the proponent attacks the $P\,\bar{t} \in \Gamma$, forcing the opponent to demonstrate $P^k\,\bar{t}^\rho$. Then the proponent can defend against $A_P\,\bar{t}$ by demonstrating the same fact, thereby winning the dialogue.

L$\bot$: The proponent attacks $\bot \in \Gamma$, leaving the opponent without any possible response and thus winning the dialogue.

L$\to$: The proponent thus attacks $\varphi \to \psi \in \Gamma$. If the opponent counters, the proponent plays according to the IH on $\Gamma \Rightarrow \varphi$. If the opponent defends, the proponent plays according to the IH on $\Gamma, \psi \Rightarrow \delta$.

R$\to$: Then the challenge is $A_\to\,\varphi\,\psi$, leaving us to prove $\text{Win}\,(k', \rho, \varphi :: \Gamma, A_\to\,\varphi\,\psi :: C)$ for some $k \le k'$. As $\varphi :: \varphi :: \Gamma \subseteq \varphi :: \Gamma$, the proponent may defend and continue playing according to the weakened inductive hypothesis.

L$\forall$: The proponent attacks $\forall\varphi \in \Gamma$ choosing $t^\rho : \mathbf{S}_k$ as the witness. One the opponent defends, the game state is $\text{Win}\,(k, t^\rho \cdot \rho, \varphi :: \Uparrow\Gamma, \uparrow(c :: C))$. By applying Theorem 20, the proponent may continue to play according to the IH as $\rho, \varphi[t] :: \Gamma \equiv_f t^\rho \cdot \rho, \varphi :: \Uparrow\Gamma$ and $\rho, c :: C \equiv_a t^\rho \cdot \rho, \uparrow(c :: C)$.

R∀: Then the challenge is $A_s\,\varphi$ for some $s : \mathbf{S}_{k'}$ and $A_s\,\varphi | k \mapsto k'$. The proponent can then defend and continue playing according to the IH on $\Uparrow\Gamma \Rightarrow \varphi$. ∎

5.2. Completeness

We show completeness the same way we do in Section 3.2. We first prove that Kripke dialogical validity entails exploding Kripke validity. We then use a prior result from Herbelin and Lee [7] to deduce completeness for the $\forall, \rightarrow, \bot$-fragment.

We extend the forcing relation to defenses as follows

$$\rho^k \Vdash D_A\,\varphi \Leftrightarrow \rho^k \Vdash \varphi \qquad \rho^k \Vdash D_W\,\varphi\,s \Leftrightarrow s \cdot \rho^k \Vdash \varphi \qquad \rho^k \Vdash D_M\,P\,\bar{t} \Leftrightarrow \rho^k \Vdash P\,\bar{t}$$

and define an auxiliary predicate on contexts Γ, k-environments ρ and challenges c

$$\Gamma \Vdash^k_\rho \bigvee \mathcal{D}_c \Leftrightarrow \forall k \le k', \alpha.\ \rho^{k'} \Vdash \Gamma \rightarrow (\forall d \in \mathcal{D}_c.\ \rho^{k'} \Vdash d \rightarrow \rho^{k'} \Vdash \alpha) \rightarrow \rho^{k'} \Vdash \alpha$$

Intuitively, $\Gamma \Vdash^k_\rho \bigvee \mathcal{D}_c$ states that Γ semantically entails the disjunction of the semantic interpretations of defenses against c under the k-environment ρ.

Lemma 21 If for some φ we have $(c :: \Gamma) \Vdash^k_\rho \bigvee \mathcal{D}_c$ for all $c \rhd \varphi$ then $\rho^k \Vdash \Gamma$ entails $\rho^k \Vdash \varphi$.

Proof Assume (1) $\forall c \rhd \varphi.\ (c :: \Gamma) \Vdash^k_\rho \bigvee \mathcal{D}_c$ and $\rho^k \Vdash \Gamma$ We proceed per case distinction on φ.

$\varphi = \bot$: As there are no defenses against $A_\bot$, choosing $\bot$ for α in (1) already yields $\rho^k \Vdash \bot$.

$\varphi = P\,\bar{t}$: We apply (1) to $A_P\,\bar{t}$.

$\varphi = \varphi \rightarrow \psi$: Let $k \le k'$ and suppose $\rho^{k'} \Vdash \varphi$. As this means $\rho^{k'} \Vdash \varphi :: \Gamma$, we may apply (1) with $A_\rightarrow\,\varphi\,\psi$.

$\varphi = \varphi \wedge \psi$: Then we can apply (1) with $A_L\,\varphi$ and $A_R\,\psi$ to obtain $\rho^k \Vdash \varphi$ and $\rho^k \Vdash \psi$, yielding $\rho^k \Vdash \varphi \wedge \psi$ overall.

$\varphi = \varphi \vee \psi$: Then we apply (1) with $A_\vee\,\varphi\,\psi$. This leaves us proving that $\rho^k \Vdash \varphi$ and $\rho^k \Vdash \psi$ each entail $\rho^k \Vdash \varphi \vee \psi$ which is clear.

$\varphi = \forall\varphi$: Let $k \le k'$ and $s : \mathbf{S}_{k'}$, we need to prove that $s \cdot \rho^{k'} \Vdash \varphi$. For this, we apply (1) with $A_s\,\varphi$.

$\varphi = \exists\varphi$: For this we apply (1) with $A_\exists\,\varphi$. This leaves us proving that for $s : \mathbf{S}_k$, $s \cdot \rho^k \Vdash \varphi$ entails $\rho^k \Vdash \exists\varphi$ which is clear. ∎

Theorem 22 Let $K, \le, \iota$ be exploding. Suppose Win $(k, \rho, \Gamma, c :: C)$ then $\Gamma \Vdash^k_\rho \bigvee \mathcal{D}_c$.

Proof We proceed per induction on $\mathrm{Win}\,(k, \rho, \Gamma, c :: C)$. Suppose $k \leq k'$ and $\rho^{k'} \Vdash \Gamma$. Now suppose that $\rho^{k'} \Vdash d$ entails $\rho^{k'} \Vdash \alpha$ for all $d \in \mathcal{D}_c$.

PA : The proponent attacks some $\varphi \in \Gamma$. We perform a case distinction on Γ.

- $\varphi = \bot$: Then $\rho^{k'} \Vdash \bot$ per assumption and thus $\rho^{k'} \Vdash \alpha$ as the structure is exploding.
- $\varphi = P\,\bar{t}$: Then we may apply the IH for the opponent defending by demonstrating $\rho^{k'} \Vdash P\,\bar{t}$ which holds per assumption.
- $\varphi = \varphi \rightarrow \psi$: We first apply Lemma 21 to the IH upon the opponent countering to obtain $\rho^{k'} \Vdash \varphi$. We can then apply the IH obtained upon the opponent admitting ψ as $\rho^{k'} \Vdash \varphi :: \Gamma$.
- $\varphi = \varphi \wedge \psi$: To apply the IH we have to demonstrate $\rho^{k} \Vdash \varphi$ or $\rho^{k} \Vdash \psi$, either of which hold as $\rho^{k} \Vdash \varphi \wedge \psi$.
- $\varphi = \varphi \vee \psi$: Then either $\rho^{k'} \Vdash \varphi$ or $\rho^{k'} \Vdash \psi$. In either case, we can apply the IH upon the opponent admitting the respective formula.
- $\varphi = \forall\varphi$: The proponent chooses some $s : \mathbf{S}_k$. As $\rho^{k'} \Vdash \forall\varphi$ we have $s \cdot \rho^{k'} \Vdash \varphi$. By applying the IH upon the opponent admitting this, we can then obtain $s \cdot \rho^{k'} \Vdash \uparrow\alpha$ meaning $\rho^{k'} \Vdash \alpha$.
- $\varphi = \varphi \vee \psi$: Then either $\rho^{k'} \Vdash \varphi$ or $\rho^{k'} \Vdash \psi$. In either case, we can apply the IH upon the opponent admitting the respective formula. with $s \cdot \rho^{k'} \Vdash \varphi$. The proof then proceeds analogously to the case of $\varphi = \forall\varphi$.

PD : The proponent thus defends against c via some $d \in \mathcal{D}_c$. We thus show $\rho^{k'} \Vdash \alpha$ by showing $\rho^{k} \Vdash d$. If $d = P_M\,P\,\bar{t}$ that means the proponent demonstrated $\rho^{k} \Vdash P\,\bar{t}$ which is transported to $\rho^{k'} \Vdash P\,\bar{t}$ via ι, meaning $\rho^{k'} \Vdash d$. In the other two cases, we may apply Lemma 21 to the IH to obtain $\rho^{k'} \Vdash d$. ■

Corollary 23 Whenever $\Gamma \models^{D} \varphi$ then also $\Gamma \models^{K} \varphi$.

Corollary 24 (Fragment Kripke completeness) When restricting to the $\forall, \rightarrow, \bot$-fragment, $\Gamma \models^{D} \varphi$ entails $\Gamma \Rightarrow \varphi$.

Proof By Corollary 23 we know $\Gamma \models^{K} \varphi$. In [7] Herbelin and Lee prove this entails $\Gamma \Rightarrow \varphi$. ■

We can also demonstrate the same relationship between Kripke material dialogues and standard Kripke validity.

Corollary 25 When restricting to the $\forall, \rightarrow, \bot$-fragment $\Gamma \models^{D} \varphi$ entailing $\Gamma \models^{KS} \varphi$ for all Γ and φ is equivalent to the Markov's principle.

Proof As $\Gamma \models^{D} \varphi$ iff $\Gamma \models^{K} \varphi$ by the previous results, $\Gamma \models^{D} \varphi$ entailing $\Gamma \models^{KS} \varphi$ is equivalent to $\Gamma \models^{K} \varphi$ entailing $\Gamma \models^{KS} \varphi$, which we have proven equivalent to the Markov's principle in [4]. ■

6. Discussion

Mechanization of active research In the course of this project, we mechanized all results from Section 3, safe for some of the corollaries, in the interactive theorem prover Coq. Mechanizing already established results in Coq is a worthy endeavor in its own right, for example yielding some insight into their computational contents when working without additional non-constructive axioms. However, we want to discuss using Coq to mechanize new results while working on them as we did here. Mechanizing the results of Section 3 revealed some mistakes in our initial definition of the rules for material dialogues which, albeit minor, broke both soundness and completeness. We missed these mistakes while working out the proofs on paper and believe they would have made it into the final report, were it not for the mechanization. Having battle tested our definitions in Section 3 gave us sufficient confidence to work on paper for the remainder of the project. It should also be noted that the mechanization took up only about a quarter of the overall time spent on the project, thanks in part due to building on top of the a large preexisting mechanization from [5]. We feel this might be a worthwhile trade off between the time requirement of a full mechanization of all results and the room for error in working solely on paper.

Proof strategies for completeness In this project, we prove completeness by transforming dialogical validity into validity in some model-based semantics and then using preexisting completeness proofs for those. The general reasoning was that this is the quickest way to obtain these completeness theorems in the framework set up by [5]. For classical material dialogues, we believe it would also be possible to obtain a direct constructive completeness proof with regards to natural deduction on the basis of a Henkin construction, although it would likely require the dialogical cut. For Kripke material dialogues, we believe we could obtain a direct constructive completeness proof for the $\forall, \rightarrow, \bot$-fragment via a normalization-by-evaluation approach, similar to that in [7]. However, we regard Kripke material validity entailing exploding Kripke validity on the full syntax as a stronger result because of its broader scope, which is why we opted for the proof strategy exhibited in this report.

Classical Kripke Dialogues The analysis of Kripke dialogues for intuitionistic first-order logic naturally brings up the question of how Kripke dialogues with a classical rule set would behave. While we chose not to pursue this question further out of time restrictions, we hold the strong belief that Kripke dialogues with classical rules should behave the same as material dialogues with classical rules. Their validity should entail classical exploding Tarski validity, as every classical exploding Tarski structure can also be viewed as an equivalent one-world Kripke structure. The more critical property is soundness: We believe that the "independence of the classicality of the structure" already demonstrated by the soundness proof for classical material dialogues should extend to even work on Kripke structures. We have, however, not checked this formally.

Benefits of Material Dialogues Another question naturally raised by this project is how material and Kripke dialogues compare to other semantics for first-order logic, especially in a constructive setting. As demonstrated in Section 3, classical material dialogues are independent of the underlying structure's classicality. This might ease work in model theory in a constructive setting. For Kripke material dialogues, there seems no apparent benefit over simply working with Kripke structures.

References

[1] Paul Lorenzen. Logik und Agon. In *Atti del XII Congresso Internazionale di Filosofia*, volume 4, pages 187–194, 1960.

[2] Paul Lorenzen. Ein dialogisches Konstruktivitätskriterium. In *Proceedings of the Symposium on Foundations of Mathematics (Warsaw, 2 – 9 September 1959)*, pages 193–200, 1961.

[3] Kuno Lorenz. *Arithmetik und Logik als Spiele*. PhD thesis, Christian-Albrechts-Universität zu Kiel, 1961.

[4] Yannick Forster, Dominik Kirst, and Dominik Wehr. Completeness theorems for first-order logic analysed in constructive type theory. In Sergei Artemov and Anil Nerode, editors, *Logical Foundations of Computer Science*, pages 47–74, Cham, 2020. Springer International Publishing.

[5] Yannick Forster, Dominik Kirst, and Dominik Wehr. Completeness theorems for first-order logic analysed in constructive type theory (extended version). *arXiv preprint arXiv:2006.04399*, 2020.

[6] Hugo Herbelin and Danko Ilik. An analysis of the constructive content of Henkin's proof of Gödel's completeness theorem. Draft, 2016.

[7] Hugo Herbelin and Gyesik Lee. Forcing-based cut-elimination for Gentzen-style intuitionistic sequent calculus. In *International Workshop on Logic, Language, Information, and Computation*, pages 209–217. Springer, 2009.

[8] Nicolaas Govert de Bruijn. Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the Church-Rosser theorem. In *Indagationes Mathematicae (Proceedings)*, volume 75, pages 381–392. Elsevier, 1972.

[9] Walter Felscher. Dialogues, strategies, and intuitionistic provability. *Annals of pure and applied logic*, 28(3):217–254, 1985.

[10] Morten Heine Sørensen and Pawel Urzyczyn. Sequent calculus, dialogues, and cut elimination. *Reflections on Type Theory, λ-Calculus, and the Mind*, pages 253–261, 2007.

A. Dialogue congruences

Given ρ, ρ' and φ, φ', we define an equivalence relation $\rho, \varphi \equiv_f \rho', \varphi'$ as below. Intuitively, $\rho, \varphi \equiv_f \rho', \varphi'$ means that φ and φ' are equal up to their terms and the pairs of terms at a certain position within these formulas are equal under evaluation in the respective environment.

$$\frac{}{\rho, \bot \equiv_f \rho', \bot} \qquad \frac{\overline{t}^{\rho} = \overline{t'}^{\rho'}}{\rho, P\,\overline{t} \equiv_f \rho', P\,\overline{t'}} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi' \quad \rho, \psi \equiv_f \rho', \psi'}{\rho, \varphi \Box \psi \equiv_f \rho', \varphi' \Box \psi'}$$

$$\frac{\forall d.\ (d \cdot \rho), \varphi \equiv_f (d \cdot \rho'), \varphi'}{\rho, \Box\varphi \equiv_f \rho', \Box\varphi'}$$

We then extend this congruence to attacks $\rho, a \equiv_a \rho', a'$, defenses $\rho, d \equiv_d \rho', d'$.

$$\frac{}{\rho, A_\bot \equiv_a \rho', A_\bot} \qquad \frac{\overline{t}^{\rho} = \overline{t'}^{\rho'}}{\rho, A_P\,\overline{t} \equiv_a \rho', A_P\,\overline{t'}} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi'}{\rho, A_L\,\varphi \equiv_a \rho', A_L\,\varphi'}$$

$$\frac{\rho, \psi \equiv_f \rho', \psi'}{\rho, A_R\,\psi \equiv_a \rho', A_R\,\psi'} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi' \quad \rho, \psi \equiv_f \rho', \psi'}{\rho, A_\vee\,\varphi\,\psi \equiv_a \rho', A_\vee\,\varphi'\,\psi'}$$

$$\frac{\rho, \varphi \equiv_f \rho', \varphi' \quad \rho, \psi \equiv_f \rho', \psi'}{\rho, A_\rightarrow\,\varphi\,\psi \equiv_a \rho', A_\rightarrow\,\varphi'\,\psi'} \qquad \frac{(s \cdot \rho), \varphi \equiv_f (s \cdot \rho'), \varphi'}{\rho, A_s\,\varphi \equiv_a \rho', A_s\,\varphi'}$$

$$\frac{\forall d.\ (d \cdot \rho), \varphi \equiv_f (d \cdot \rho'), \varphi'}{\rho, A_\exists\,\varphi \equiv_a \rho', A_\exists\,\varphi'}$$

$$\frac{\overline{t}^{\rho} = \overline{t'}^{\rho'}}{\rho, D_M\,P\,\overline{t} \equiv_d \rho', D_M\,P\,\overline{t'}} \qquad \frac{\rho, \varphi \equiv_f \rho', \varphi'}{\rho, D_A\,\varphi \equiv_d \rho', D_A\,\varphi'} \qquad \frac{(s \cdot \rho), \varphi \equiv_f (s \cdot \rho'), \varphi'}{\rho, D_W\,\varphi\,s \equiv_d \rho', D_W\,\varphi'\,s}$$

We extend all of the above to lists with

$$\frac{}{\rho, [] \equiv_x \rho', []} \qquad \frac{\rho, a \equiv_x \rho', a' \quad \rho, A \equiv_x \rho', A'}{\rho, (a :: A) \equiv_x \rho', (a' :: A')}$$

Lastly we define a congruence between substitutions $\rho, \sigma \equiv_s \rho', \sigma'$ which holds if for all variables x we have that $(\sigma\,x)^{\rho} = (\sigma'\,x)^{\rho'}$. We now state all of the properties required of the relations to show Lemma 1. All of them have been proven in the Coq mechanization accompanying this report.

Fact 26

1. $\rho, \varphi \equiv_f \rho', \varphi'$ is an equivalence relation
2. $\rho, \varphi \equiv_a \rho', \varphi'$ is an equivalence relation
3. $\rho, \varphi \equiv_d \rho', \varphi'$ is an equivalence relation
4. $\rho, \varphi \equiv_f \rho', \varphi'$ and $a \rhd \varphi$ mean there is a $a' \rhd \varphi'$ with $\rho, a \equiv_a \rho', a'$
5. If $\rho, a \equiv_a \rho', a'$ then if $\text{adm}\, a = \ulcorner\varphi\urcorner$ then $\text{adm}\, a' = \ulcorner\varphi'\urcorner$ such that $\rho, \varphi \equiv_f \rho', \varphi'$
6. $\rho, a \equiv_a \rho', a'$ and $d \in \mathcal{D}_a$ mean there is a $d' \in \mathcal{D}_{a'}$ with $\rho, d \equiv_d \rho', d'$
7. If $\rho, d \equiv_d \rho', d'$ and ρ justifies d then ρ' justifies d'
8. If $\rho, a \equiv_a \rho', a'$ and $\rho, A \equiv_f \rho', A'$ then $\rho, (a :: A) \equiv_f \rho', (a' :: A')$
9. $\rho, \sigma \equiv_s \rho', \sigma'$ means $\rho, \varphi[\sigma] \equiv_f \rho', \varphi[\sigma']$
10. $\rho, \sigma \equiv_s \rho', \sigma'$ means $\rho, a[\sigma] \equiv_a \rho', a[\sigma']$