NBE with constants

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Proof and term normalization is an important tool of proof theory as it yields unique representations that are usually more convenient to work with. One method of obtaining such a normalization procedure was put forward by Berger and Schwichtenberg [1], dubbed "normalization by evaluation". Since their seminal paper, a lot of work has gone towards streamlining and extending their method, often in a categorical framework. While extensions of the method often require fairly complicated arguments, Berger and Schwichtenberg had themselves provided an elegant means of extension which didn't get accounted for in later, more categorical renderings. We thus demonstrate how to adapt this extension method to categorical NBE approaches, such as that of Altenkirch et al [2].

1 Syntax

We work in the simply typed lambda calculus. Types τ are build up from a base type o and function types $\tau \to \tau$. We take C to be a set of typed constants $c : \tau$. We use **intrinsically typed syntax** with DeBruijn-style binders as defined below. Contexts Γ are finite lists of types.

$$\textsc{Id}\,\frac{}{\Gamma, \tau \vdash 0 : \tau} \qquad \textsc{Skip}\,\frac{\Gamma \vdash n : \tau}{\Gamma, \sigma \vdash n+1 : \tau} \qquad \textsc{Abs}\,\frac{\Gamma, \sigma \vdash s : \tau}{\Gamma \vdash \lambda s : \sigma \to \tau}$$

$$\textsc{App}\,\frac{\Gamma \vdash s : \sigma \to \tau \qquad \Gamma \vdash t : \sigma}{\Gamma \vdash st : \tau} \qquad \textsc{Cons}\,\frac{c : \tau \in C}{\Gamma \vdash c : \tau}$$

Given a transition relation $\succ$ for terms involving constants from C, we define **η-long normal forms** as given below.

$$\textsc{Var}\ \frac{\Gamma \vdash i : \tau_0 \to \dots \to \tau_n \to o \quad \Gamma \vdash_{\mathrm{NF}} s_0 : \tau_0 \ \dots \ \Gamma \vdash_{\mathrm{NF}} s_n : \tau_n}{\Gamma \vdash_{\mathrm{NF}} i\, s_0 \dots s_n : o}$$

$$\textsc{Con}\ \frac{c : \tau_0 \to \dots \to \tau_n \to o \in C \quad \Gamma \vdash_{\mathrm{NF}} s_0 : \tau_0 \ \dots \ \Gamma \vdash_{\mathrm{NF}} s_n : \tau_n \quad c\, s_0 \dots s_n \not\succ t}{\Gamma \vdash_{\mathrm{NF}} c\, s_0 \dots s_n : o}$$

$$\textsc{Fun}\ \frac{\Gamma, \sigma \vdash_{\mathrm{NF}} s : \tau}{\Gamma \vdash_{\mathrm{NF}} \lambda s : \sigma \to \tau}$$

Lastly, we define **substitutions** between contexts.

$$\textsc{Base}\ \frac{}{\Gamma \vdash \diamond : \diamond} \qquad \textsc{Cons}\ \frac{\Gamma \vdash \theta : \Delta \quad \Gamma \vdash s : \tau}{\Gamma \vdash \theta, s : \Delta, \tau}$$

We call a substitution $\Gamma \vdash \theta : \Delta$ a **renaming** if it is comprised of variables. Given a renaming $\Gamma \vdash \theta : \Delta$ we can obtain a new renaming $\Gamma, \tau \vdash \uparrow\theta : \Delta$ by shifting each index in θ by one. With this, we may define an operation which given a term $\Delta \vdash s : \tau$ and a renaming $\Gamma \vdash \theta : \Delta$ produces a renamed term $\Gamma \vdash s[\theta] : \tau$ as follows:

$$\Gamma \vdash i[\theta] : \tau = \Gamma \vdash \theta_i : \tau \qquad \Gamma \vdash (s\, t)[\theta] : \tau = \Gamma \vdash (s[\theta])\,(t[\theta]) : \tau$$

$$\Gamma \vdash c[\theta] : \tau = \Gamma \vdash c : \tau \qquad \Gamma \vdash (\lambda\, s)[\theta] : \sigma \to \tau = \Gamma \vdash \lambda\,(s[\Uparrow\theta, 0]) : \sigma \to \tau$$

With this, we can define an operation $\Gamma, \tau \vdash \Uparrow\theta : \Delta$ which applies the renaming $\Gamma, \tau \vdash \uparrow\mathrm{id} : \Gamma$ to every term of θ. With this, in turn, we can define a full substitution operation via the equations above.

2 Berger & Schwichtenberg

2.1 λ-models

We present a version of [1] adapted to our DeBruijn syntax given above.

A **λ-model** M consists of a family of sets indexed by the types M^τ and for each constant $c : \tau \in C$ a choice of value $c_M \in M^\tau$. Further, there is an operation $\cdot : M^{\sigma\to\tau} \times M^\sigma \to M^\tau$ and a **congruence relation** $\equiv_\tau \subseteq M^\tau \times M^\tau$ which captures extensionality

$$s \equiv_{\sigma\to\tau} s' \wedge t \equiv_\sigma t' \implies st \equiv_\tau s't' \qquad (\forall t.\ st \equiv_\tau s't) \implies s \equiv_{\sigma\to\tau} s'$$

We write $\equiv$ if τ is clear from the context. Now, given a context Γ, we define $\textsc{Env}(\Gamma) := \{\gamma : |\Gamma| \to \bigcup_\tau M^\tau \mid \forall i < |\Gamma|.\ \gamma(i) \in M^{\Gamma_i}\}$. Lastly, we require an **interpretation function** such that for $\Gamma \vdash s : \tau$ and $\gamma \in \textsc{Env}(\Gamma)$ we have $|s|\gamma \in M^\tau$ satisfying

$$|i|\gamma \equiv \gamma(i) \qquad |c|\gamma \equiv c_M \qquad |st|\gamma \equiv |s|\gamma \cdot |t|\gamma \qquad \forall m.\ |\lambda s|\gamma \cdot m \equiv |s|(\gamma, m)$$

Furthermore, we require that for any $c\, s_0 \dots s_n \succ t$ we have $|c\, s_0 \dots s_n|\gamma \equiv |t|\gamma$.

2.2 Normalization by evaluation

We work in models in which M^o contains "enough syntactic material" to represent the terms of our calculus as well as some operation on them. For this, we extend C with constants $\hat{\lambda} : o \to o$, $\mathrm{app} : o \to o \to o$, $0 : o$, $S : o \to o$, $\mathrm{swap} : o \to o \to o \to o$ and for each constant $c : \tau \in C$ a new constant $\hat{c} : o$. We then call a model **admissible** if its congruence satisfies the following

$$\mathrm{swap}\,(\mathrm{app}\,s\,t)\,n\,m \equiv \mathrm{app}\,(\mathrm{swap}\,s\,n\,m)\,(\mathrm{swap}\,t\,n\,m)$$

$$\mathrm{swap}\,(\hat{\lambda}\,s)\,n\,m \equiv \hat{\lambda}\,(\mathrm{swap}\,s\,(S\,n)\,(S\,m)) \qquad \mathrm{swap}\,(\hat{c}\,n\,m) \equiv \hat{c}$$

$$\mathrm{swap}\,i\,n\,m \equiv \begin{cases} S\,i & \text{if } m \leq i < n \\ m & \text{if } i = n \\ i & \text{otherwise} \end{cases}$$

Intuitively, swap can be used to "pull variables forward". For example, if $\sigma, \Gamma \vdash s : \tau$ and $t \equiv \mathrm{swap}\,s\,|\Gamma|\,0$ then we would have $\Gamma, \sigma \vdash t : \tau$. It is easy to see that we can syntactically embed any $\Gamma \vdash s : \tau$ into any admissible model using the "syntactic material" above, which we denote by $\lceil \Gamma \vdash s : \tau \rceil$.

We now move on to giving the two functions needed for normalization by evaluation: a quoting function $q^\tau_\Gamma : \tau \to o$ and an unquoting function $u^\tau_\Gamma : o \to \tau$. We inductively define these as terms of our extended lambda calculus.

$$\begin{aligned} q^\tau_\Gamma &: \tau \to o \\ q^o_\Gamma\,t &= t \\ q^{\sigma\to\tau}_\Gamma\,f &= \hat{\lambda}\,(\mathrm{swap}\,(q^\tau_{\sigma,\Gamma}\,(f\,(u^\sigma_{\sigma,\Gamma}|\Gamma|)))\,|\Gamma|\,0) \end{aligned} \qquad \begin{aligned} u^\tau_\Gamma &: o \to \tau \\ u^o_\Gamma\,t &= t \\ u^{\sigma\to\tau}_\Gamma\,t\,m &= u^\tau_\Gamma\,(\mathrm{app}\,t\,(q^\sigma_\Gamma\,m)) \end{aligned}$$

As the above definitions were all within the extended calculus, we may interpret them into any admissible λ-model M. For any renaming $\Gamma \vdash \theta : \Delta$ we may thus define a $\hat{\theta} \in \textsc{Env}(\Delta)$ with $\hat{\theta}_i = |u^{\Delta_i}_\Gamma| \cdot \lceil \Gamma \vdash \theta_i : \Delta_i \rceil$ (note that no environment is needed for $|u^{\Delta_i}_\Gamma|$ as it is closed). For readability, we from now on omit the $| - |$ for quoting and unquoting as well as the explicit λ-model application operation. We can now state the main lemma:

Lemma 1 Let M be an admissible model such that for any $\Gamma \vdash_{\mathrm{NF}} c\,s_0 \ldots s_n : o$ we have that $|c\,s_0 \ldots s_n|\hat{\theta} \equiv u^o_\Gamma(|\Delta \vdash_{\mathrm{NF}} c\,s_0 \ldots s_n : o|\hat{\theta})$. For any $\Delta \vdash_{\mathrm{NF}} r : \tau$ and $\Gamma \vdash \theta : \Delta$ we have that

$$q^\tau_\Gamma\,(|\Delta \vdash_{\mathrm{NF}} r : \tau|\hat{\theta}) \equiv \lceil \Gamma \vdash_{\mathrm{NF}} r[\theta] : \tau \rceil$$

Proof We show this claim per induction on the normal form r. There are three cases we need to consider:

VAR: Then $r = i\, s_0 \dots s_n$ for $\Delta \vdash i : \tau_0 \to \dots \to \tau_n \to o$ and $\Delta \vdash_{\mathrm{NF}} s_i : \tau_i$. Pick some $\Gamma \vdash \theta : \Delta$.

$$\begin{aligned}
& q^o_\Gamma(|\Delta \vdash_{\mathrm{NF}} i\, s_0 \dots s_n : o|\hat{\theta}) \\
\equiv\ & |\Delta \vdash_{\mathrm{NF}} c\, s_0 \dots s_n : o|\hat{\theta} \\
\equiv\ & (|\Delta \vdash i : \tau_0 \to \dots \to \tau_n \to o|\hat{\theta})\ (|\Delta \vdash_{\mathrm{NF}} s_0 : \tau_0|\hat{\theta}) \dots (|\Delta \vdash_{\mathrm{NF}} s_n : \tau_n|\hat{\theta}) \\
\equiv\ & (u^{\tau_0 \to \dots \to \tau_n \to o}_\Gamma\ i)\ (|\Delta \vdash_{\mathrm{NF}} s_0 : \tau_0|\hat{\theta}) \dots (|\Delta \vdash_{\mathrm{NF}} s_n : \tau_n|\hat{\theta}) \\
\overset{*}{\equiv}\ & \mathrm{app}\, \lceil \Gamma \vdash i : \tau_0 \to \dots \to \tau_n \to o \rceil\ (q^{\tau_0}_\Gamma |\Delta \vdash_{\mathrm{NF}} s_0 : \tau_0|\hat{\theta}) \dots (q^{\tau_n}_\Gamma |\Delta \vdash_{\mathrm{NF}} s_n : \tau_n|\hat{\theta}) \\
\overset{\mathrm{IH}}{\equiv}\ & \mathrm{app}\, \lceil \Gamma \vdash i : \tau_0 \to \dots \to \tau_n \to o \rceil\ \lceil \Gamma \vdash_{\mathrm{NF}} s_0[\theta] : \tau_0 \rceil \dots \lceil \Gamma \vdash_{\mathrm{NF}} s_n[\theta] : \tau_n \rceil \\
\equiv\ & \lceil \Gamma \vdash_{\mathrm{NF}} (i\, s_0 \dots s_n)[\theta] : o \rceil
\end{aligned}$$

where in $(*)$ is a repeated unfolding of the definition of the $u^{\sigma \to \tau}_\Gamma$-case.

CON: Then $r = c\, s_0 \dots s_n$ for $c : \tau_0 \to \dots \to \tau_n \to o \in \mathcal{C}$, $\Delta \vdash_{\mathrm{NF}} s_i : \tau_i$. Pick some $\Gamma \vdash \theta : \Delta$.

$$\begin{aligned}
& q^o_\Gamma(|\Delta \vdash_{\mathrm{NF}} c\, s_0 \dots s_n : o|\hat{\theta}) \\
\equiv\ & |\Delta \vdash_{\mathrm{NF}} c\, s_0 \dots s_n : o|\hat{\theta} \\
\overset{*}{\equiv}\ & (u^{\tau_0 \to \dots \to \tau_n \to o}_\Gamma\ \hat{c})\ (|\Delta \vdash_{\mathrm{NF}} s_0 : \tau_n|\hat{\theta}) \dots (|\Delta \vdash_{\mathrm{NF}} s_n : \tau_n|\hat{\theta}) \\
\equiv\ & \mathrm{app}\, \lceil \Gamma \vdash_{\mathrm{NF}} c : \tau_0 \to \dots \to \tau_n \to o \rceil\ (q^{\tau_0}_\Gamma |\Delta \vdash_{\mathrm{NF}} s_0 : \tau_0|\hat{\theta}) \dots (q^{\tau_n}_\Gamma |\Delta \vdash_{\mathrm{NF}} s_n : \tau_n|\hat{\theta}) \\
\overset{\mathrm{IH}}{\equiv}\ & \mathrm{app}\, \lceil \Gamma \vdash_{\mathrm{NF}} c : \tau_0 \to \dots \to \tau_n \to o \rceil\ \lceil \Gamma \vdash_{\mathrm{NF}} s_0[\theta] : \tau_0 \rceil \dots \lceil \Gamma \vdash_{\mathrm{NF}} s_n[\theta] : \tau_n \rceil \\
\equiv\ & \lceil \Gamma \vdash_{\mathrm{NF}} (c\, s_0 \dots s_n)[\theta] : o \rceil
\end{aligned}$$

where $(*)$ is precisely the assumption we made about the admissible model M.

FUN: Then $r = \lambda s$ for a $\Delta, \sigma \vdash_{\mathrm{NF}} s : \tau$. Pick some $\Gamma \vdash \theta : \Delta$.

$$\begin{aligned}
& q^{\sigma \to \tau}_\Gamma(|\Delta \vdash_{\mathrm{NF}} \lambda s : \sigma \to \tau|\hat{\theta}) \\
\equiv\ & \hat{\lambda}\, (\mathrm{swap}\, (q^\tau_{\sigma,\Gamma}\, ((|\Delta \vdash_{\mathrm{NF}} \lambda s : \sigma \to \tau|\hat{\theta})\ (u^\sigma_{\sigma,\Gamma}\, |\Gamma|)))\ |\Gamma|\ 0) \\
\equiv\ & \hat{\lambda}\, (\mathrm{swap}\, (q^\tau_{\sigma,\Gamma}\, (|\Delta, \sigma \vdash_{\mathrm{NF}} s : \tau|(\hat{\theta}, u^\sigma_{\sigma,\Gamma}\, |\Gamma|)))\ |\Gamma|\ 0) \\
\overset{*}{\equiv}\ & \hat{\lambda}\, (\mathrm{swap}\, (q^\tau_{\sigma,\Gamma}\, (|\Delta, \sigma \vdash_{\mathrm{NF}} s : \tau|(\widehat{\theta, |\Gamma|})))\ |\Gamma|\ 0) \\
\overset{\mathrm{IH}}{\equiv}\ & \hat{\lambda}\, (\mathrm{swap}\, \lceil \sigma, \Gamma \vdash_{\mathrm{NF}} s[\theta, |\Gamma|] : \tau \rceil\ |\Gamma|\ 0) \\
\equiv\ & \hat{\lambda}\, \lceil \Gamma, \sigma \vdash_{\mathrm{NF}} s[\theta, 0] : \tau \rceil \\
\equiv\ & \lceil \Gamma \vdash_{\mathrm{NF}} (\lambda s)[\theta] : \tau \rceil
\end{aligned}$$

in $(*)$ we use that $\sigma, \Gamma \vdash \theta, |\Gamma| : \Delta, \sigma$ and that $u^{\Gamma_i}_{\sigma,\Gamma}\, i \equiv u^{\Gamma_i}_\Gamma\, i$ for $i < |\Gamma|$. ∎

From the lemma above, we can now easily derive our actual claim:

Theorem 2 Given an admissible λ-model M satisfying the additional condition on constants from Lemma 1 then for any $\Delta \vdash s : \tau$ and $\Gamma \vdash \theta : \Delta$ we have

$$q^{\tau}_{\Gamma}\, |\Delta \vdash s : \tau|\hat{\theta} \equiv \ulcorner \Gamma \vdash_{\mathrm{NF}} r[\theta] : \tau \urcorner$$

where r is the normal form of s.

Proof For this, observe that per the equations required of $| - |-$ above we know that $|\Delta \vdash s : \tau|\hat{\theta} \equiv |\Delta \vdash_{\mathrm{NF}} r : \tau|\hat{\theta}$ and thus by Lemma 1 that $q^{\tau}_{\Gamma}\, |\Delta \vdash s : \tau|\hat{\theta} \equiv q^{\tau}_{\Gamma}\, |\Delta \vdash_{\mathrm{NF}} r : \tau|\hat{\theta} \equiv \ulcorner \Gamma \vdash_{\mathrm{NF}} r[\theta] : \tau \urcorner$. ∎

2.3 Interpreting constants

So far, we have left the model fully abstract. We now construct one suitable model and explore to what extend we can construct suitable constant interpretations. For this, we choose $C = \{z : o, s : o \to o\}$. We take M^o to be the set of all untyped terms of this calculus and $M^{\sigma \to \tau} = \{f : M^{\sigma} \to M^{\tau}\}$. Lastly, we take $z_M = \hat{z}$ and $s_m = \hat{s}$. It is easy to see that this model is an admissible model for the C-calculus.

However, it does not seem clear how to interpret more involved constants, say a recursor $R : o \to (o \to o) \to o \to o$. The interpretation Berger and Schwichtenberg [1] propose for it is

$$R^{\tau}_{M}\, a\, f\, n = \begin{cases} a & n = \hat{z} \\ f\, (R^{\tau}_{M}\, a\, f\, n') & n = \hat{s}\, n' \\ u^{\tau \to (\tau \to \tau) \to o \to \tau}\, \hat{R}^{\tau}\, a\, f\, n & \text{otherwise} \end{cases}$$

however, differing from their presentation, we need to choose a suitable Γ for the unquoting in the last line, accounting for the used variables in a and f. This does not seem possible as a and f do not carry "enough" information. Again, this seems to be an issue with the naïvety of our models.

3 Synthesis

Seeing how the naïve models of Berger and Schwichtenberg do not seem suitable for the normalization of interesting constants in DeBruijn syntax, we perform a synthesis between their work and that of Altenkirch et al. [2]. The richer presheaf structure of their approach eases the "context guessing" that hindered our DeBruijn adaption in Section 2.2.

3.1 Presheaf λ-models

Substitutions as laid out in Section 1 form the **category** **Subst**. Its objects are contexts and morphisms $\theta : \Gamma \to \Delta$ are substitutions $\Gamma \vdash \theta : \Delta$. It is easy to define suitable identities and composition operations and to check that this indeed forms a category.

Similar to Altenkirch et al. [2] we work in the **category of presheaves over $\widehat{\text{Ren}}$**, the subcategory Subst restricted to renamings. For a base type interpretation $F \in \widehat{\text{Ren}}$ we define a **semantic embedding** $[\![\tau]\!]$ for types into $\widehat{\text{Ren}}$:

$$[\![o]\!] = F \qquad\qquad [\![\sigma \to \tau]\!] = [\![\sigma]\!] \Rightarrow [\![\tau]\!]$$

this can be extended to contexts $\Gamma = \tau_n, ..., \tau_0$ with $\text{Env}(\Gamma) = [\![\tau_n]\!] \times ... \times [\![\tau_0]\!]$. We write $\gamma[a] \in \text{Env}(\Gamma, \tau)_\Delta$ for the extension of $\gamma \in \text{Env}(\Gamma)_\Delta$ with $a \in [\![\tau]\!]_\Delta$.

If we fix for each constant $c : \tau \in C$ an interpretation $c^F \in [\![\tau]\!]_\diamond$, we obtain a full functor $\text{Env} : \text{Subst} \to \widehat{\text{Ren}}$ which is defined as below on a substitution $\Gamma \vdash s : \tau$ and extended to bigger codomains in the obvious way.

$$\begin{aligned}
\text{Env}(\Gamma \vdash c : \tau)_\Delta(\gamma) &= c^F \cdot \Delta \\
\text{Env}(\Gamma \vdash i : \tau)_\Delta(\gamma) &= \gamma(i) \\
\text{Env}(\Gamma \vdash \lambda s : \sigma \to \tau)_\Delta(\gamma) &= (\theta, a)_{\Delta'} \mapsto \text{Env}(\Gamma, \sigma \vdash s : \tau)_{\Delta'}((\gamma \cdot \theta)[a]) \\
\text{Env}(\Gamma \vdash s\, t : \tau)_\Delta(\gamma) &= (\text{Env}(\Gamma \vdash s : \sigma \to \tau)_\Delta(\gamma)(\text{id}, \text{Env}(\Gamma \vdash t : \sigma)_\Delta(\gamma))
\end{aligned}$$

where we denote the weakening $\Delta \to \diamond$ by Δ in the first line. In accordance with Section 2 we write $|\Gamma \vdash s : \tau|\gamma$ for $\text{Env}(\Gamma \vdash s : \tau)_\Delta(\gamma)$ when the Δ is clear from the context.

3.2 A naïve adaption

We begin by constructing a very faithful categorical adaption of [1] which is a little less elegant than that of Altenkirch et al. [2].

For our interpretation, we define two important presheaves:

$$\begin{aligned}
\text{Tm}^\tau_\Gamma &= \{s \mid \Gamma \vdash s : \tau\} & s \cdot \theta &= s[\theta] & &\text{for } s \in \text{Tm}^\tau_\Gamma, \theta : \Delta \to \Gamma \\
\text{Lt}^\tau_\Gamma &= \{i \mid \Gamma \vdash i : \tau\} \cup \{c \mid c : \tau \in C\} & s \cdot \theta &= s[\theta] & &\text{for } s \in \text{Lt}^\tau_\Gamma, \theta : \Delta \to \Gamma
\end{aligned}$$

intuitively, Tm^τ_Γ contains all terms of type τ under Γ whereas Lt^τ_Γ contains only the "letters" of type τ under Γ.

Now we fix our base type interpretation to be Tm^o define quoting and unquoting functions, similar to Altenkirch et al:

$$\begin{aligned}
q^\tau &: [\![\tau]\!] \to \text{Tm}^\tau & u^\tau &: \text{Lt}^\tau \to [\![\tau]\!] \\
q^o_\Gamma &= \text{id} & u^o_\Gamma &= \text{id} \\
q^{\sigma\to\tau}_\Gamma f &= \lambda\,(q^\tau_{\Gamma,\sigma}\,(f\,(\Uparrow\!\text{id}, u^\sigma_{\Gamma,\sigma}\,0))) & u^{\sigma\to\tau}_\Gamma s &= (\theta, a)_\Delta \mapsto u^\tau_\Delta\,((s \cdot \theta)\,(q^\sigma_\Delta\, a))
\end{aligned}$$

Note that when comparing our functions to those of Altenkirch et al, we have strengthened the domain of u^τ to Lt^τ and thus had to weaken the codomain of q^τ to Tm^τ.

Now we can carry out a proof of Lemma 1 in this setting in an analogous manner.

Lemma 3 Let c^F be chosen such that for any $\Gamma \vdash_{\mathrm{NF}} c\, s_1 \ldots s_n : o$ and $\theta : \Delta \to \Gamma$

$$|\Gamma \vdash c\, s_1 \ldots s_n : o|\hat{\theta} = (u^{\Delta}_{\tau_1 \to \ldots \to \tau_n \to o}\, c)(\mathrm{id}, |\Gamma \vdash s_1 : \tau_1|\hat{\theta}) \ldots (\mathrm{id}, |\Gamma \vdash s_n : \tau_n|\hat{\theta})$$

then we have that for any $\Gamma \vdash_{\mathrm{NF}} s : \tau$ and $\theta : \Delta \to \Gamma$

$$q^{\Delta}_{\tau}\,(|\Gamma \vdash s : \tau|\hat{\theta}) = \Delta \vdash s[\theta] : \tau$$

Proof Analogous to that of Lemma 1. ■

To demonstrate the strength of this result, we give interpretations suitable for normalizing Gödel's T. Thus, we take

$$C = \{z : o, s : o \to o\} \cup \{R_\tau : \tau \to (\tau \to \tau) \to o \to \tau \mid \tau \text{ a type}\}$$

and as the interpretations $z^F = z$ and $s^F = u^{o \to o}_{\diamond}\, s$. For the recursor, we define

$$R^F_\tau(\theta, a)(\theta', f)(\theta'', n) = \begin{cases} a \cdot \theta' \cdot \theta'' & \text{if } n = z \\ f(\theta'', R^F_\tau(\theta, a)(\theta', f)(\theta'', n')) & \text{if } n = s\, n' \\ u^{\tau \to (\tau \to \tau) \to o \to \tau}_{\diamond}\, R_\tau(\theta, a)(\theta', f)(\theta'', n) & \text{otherwise} \end{cases}$$

To verify that this definition is indeed suitable for Lemma 3, note that $\Gamma \vdash_{\mathrm{NF}} R_\tau\, a\, f\, n : \tau$ means that n is not a Peano numeral, as the expression would not be normal otherwise, meaning the third case applies as desired.

3.3 A full adaption

In the last section, we gave a faithful categorical adaption of Berger and Schwichtenberg [1]. For this, we extended the domain of u^{τ}_{Γ} to $\mathrm{L}\tau^{\tau}_{\Gamma}$ to be able to state the invariant on $\Gamma \vdash_{\mathrm{NF}} c\, s_1 \ldots s_n : o$ used to prove Lemma 3. As a side effect of this, we lost the elegantly tight characterization of the types of q^{τ}_{Γ} and u^{Γ}_{τ} as we could not guarantee normality anymore. However, careful analysis of q^{τ}_{Γ} and u^{Γ}_{τ} as well as the proofs of Lemmas 1 and 3 reveal that this is not necessary. Indeed, Berger and Schwichtenberg only apply u^{Γ}_{τ} to the constants in their invariant as convenient shorthand, what one really requires is that

$$|\Gamma \vdash c\, s_1 \ldots s_n : o|\hat{\theta} = \Gamma \vdash c\, (q^{\tau_1}_{\Delta}\, |\Gamma \vdash s_1 : \tau_1|\hat{\theta}) \ldots (q^{\tau_n}_{\Delta}\, |\Gamma \vdash s_n : \tau_n|\hat{\theta}) : o$$

for the proof to go through.

For certain choices of constants and reductions on them, this can be achieved in a presentation fully analogous to that of Altenkirch et al. [2], which we demonstrate in this section. We begin by defining a new notion of normal forms:

$$\frac{}{\Gamma \vdash_{\mathrm{ANE}} i : \tau} \qquad \frac{\Gamma \vdash_{\mathrm{ANE}} s : \tau}{\Gamma \vdash_{\mathrm{ANF}} s : \tau} \qquad \frac{\Gamma \vdash_{\mathrm{ANE}} s : \sigma \to \tau \quad \Gamma \vdash_{\mathrm{ANF}} t : \sigma}{\Gamma \vdash_{\mathrm{ANE}} s\, t : \tau}$$

$$\textsc{Con}\ \frac{c : \tau_0 \to \ldots \to \tau_n \to o \in C \quad \Gamma \vdash_{\mathrm{ANF}} s_0 : \tau_0 \ \ldots\ \Gamma \vdash_{\mathrm{ANF}} s_n : \tau_n \quad c\, s_0 \ldots s_n \not\succ t}{\Gamma \vdash_{\mathrm{ANF}} c\, s_0 \ldots s_n : o}$$

$$\frac{\Gamma, \sigma \vdash_{\mathrm{ANF}} s : \tau}{\Gamma \vdash_{\mathrm{ANF}} \lambda s : \sigma \to \tau}$$

If normality is preserved under renaming, we obtain presheaves $\textsc{Anf}^\tau$ and $\textsc{Ane}^\tau$. Note that this rules out transition rules such as $c\, x\, x \succ c'$. If we take $\textsc{Anf}^o$ as the base type then the definitions from Section 3.2 yield functions $q^\tau : [\![\tau]\!] \to \textsc{Anf}^\tau$ and $u^\tau : \textsc{Ane}^\tau \to [\![\tau]\!]$. With this, we can state our modified lemma.

Lemma 4 Let c^F be chosen such that for any $\Gamma \vdash_{\mathrm{NF}} c\, s_1 \ldots s_n : o$ and $\theta : \Delta \to \Gamma$

$$|\Gamma \vdash c\, s_1 \ldots s_n : o|\hat{\theta} = \Gamma \vdash c\, (q^{\tau_1}_\Delta\, |\Gamma \vdash s_1 : \tau_1|\hat{\theta}) \ldots (q^{\tau_n}_\Delta\, |\Gamma \vdash s_n : \tau_n|\hat{\theta}) : o$$

then we have that for any $\Gamma \vdash_{\mathrm{NF}} s : \tau$ and $\theta : \Delta \to \Gamma$

$$q^\Delta_\tau\, (|\Gamma \vdash s : \tau|\hat{\theta}) = \Delta \vdash s[\theta] : \tau$$

Proof Analogous to Lemmas 1 and 3. ■

Now it remains to demonstrate that this is formulation is actually applicable. For this, we revisit the C capturing Gödel's T. This time, we define $z^F = z$ and $s^F(\theta, n) = s\, n$. For the recursor, note that any type $\tau = \sigma_1 \to \ldots \to \sigma_n \to o$ and we thus may change the last case of $R^F_\tau(\theta, a)_\Delta(\theta', f)_{\Delta'}(\theta'', n)_{\Delta''}$ to

$$\begin{aligned}(\theta_1, s_1)_{\Delta_1} \ldots (\theta_n, s_n)_{\Delta_n} \mapsto R_\tau\, &(q^\tau_{\Delta_n}\, (a \cdot (\theta_n \circ \ldots \circ \theta_1 \circ \theta'' \circ \theta')))\\ &(q^{\tau \to \tau}_{\Delta_n}\, (f \cdot (\theta_n \circ \ldots \circ \theta_1 \circ \theta'')))\\ &(q^o_{\Delta_n}\, (n \cdot (\theta_n \circ \ldots \circ \theta_1)))\\ &(q^{\sigma_1}_{\Delta_n}\, (s_1 \cdot (\theta_n \circ \ldots \circ \theta_1))) \ldots (q^{\sigma_n}_{\Delta_n}\, s_n)\end{aligned}$$

The interpretations yield normal values and are thus well-defined by the Con rule.

4 Discussion

4.1 Insights

In this section, we gather some of the insights into the normalization-by-evaluation technique and some of the subtler points of its definitions.

For Berger and Schwichtenberg's models, the interpretation function has to be provided by the model as we do not require models to supply a semantic abstraction operation or similar means which we would need to define $|\lambda s|\gamma$ ourselves.

The "tricky" part of defining $q_\Gamma^{\sigma\to\tau}$ is picking the fresh variable to evaluate the semantic function on. We compare our solution in Section 2.2 to this problem to that of [1] and [2].

- Berger and Schwichtenberg [1] work with named binders instead of DeBruijn indexes. They thus define q_Γ^τ and u_Γ^τ on α-equivalence classes instead of plain terms as in [2] and our presentation. Their formalization of these classes means they are given an explicit variable name in $q_\Gamma^{\sigma\to\tau}$ that is guaranteed to not clash with variables used in the semantic function, thus freeing them of having to make the choice themselves.
- Altenkirch et al. [2] work in the presheaf category over weakenings/renamings as opposed to the more naïve λ-models used by [1] and us. That means they can simply weaken the semantic function with the weakening $w : \Gamma, \sigma \to \Gamma$, thereby ensuring that the variable $\Gamma, \sigma \vdash 0 : \sigma$ is not used.
- We sadly cannot reuse either of the previous approaches. The approach of Berger and Schwichtenberg is not applicable as terms with DeBruijn indexes are another way of representing α-equivalence classes of named binders which does not yield the "free choice" that their approach gives. The approach by Altenkirch et al. crucially relies on the fact that their model has an underlying "weakening structure" which our naïve λ-models do not offer.

  Our approach can non the less be seen as an adaption of that of Altenkirch et al. as we, too, define q_Γ^τ and u_Γ^τ not only indexed by types but also by contexts Γ. This context Γ is precisely the variables used by the semantic term $|\Delta \vdash s : \sigma \to \tau|\hat{\theta}$ which means we may extend it "upwards" into σ, Γ without the need to shift the already assigned variables in $\hat{\theta}$. The downside of this approach is that we need to "swap back" the σ, Γ into Γ, σ via the swap function to ensure that $\hat{\lambda}$ binds the correct variable.

As in the categorical settings, the precise definition of $\Gamma \vdash_{\mathrm{NF}} r : \tau$ is important to the proof in Section 2.2. In this case, it means that in Lemma 1 we only need to consider one case of Φ_τ for each case of the normal form definition, leaving us with very few case-distinctions overall. The same can be observed in Section 3.3 where adding Con rule to the definition of normal-forms enables us to carry out the proof.

4.2 Future Work

In this work, we have adapted the proof strategy of Berger and Schwichtenberg [1] to the models of Altenkirch et al. [2]. Another question worth exploring would be if the proof strategy of Altenkirch et al, dubbed "twisted gluing", can be extended to constants in a similar manner. Twisted gluing works with a contextual CCC whose objects are pairs of syntactical types and presheaves with a quoting and unquoting morphism between them satisfying some equations. Importantly, the presheaf-part of the exponentials of that category are sub-presheaves of the usual presheaf exponentials. It seems that to extend this approach to constants, it would suffice to show that the constant interpretations lie within this new category. However, as the condition on twisted exponentials is given in terms of all members of the domain, there seems no simple way of translating the condition of Berger and Schwichtenberg, which is only concerned with the members of the presheaf that are interpretations of λ-terms, to this setting.

References

[1] Ulrich Berger and Helmut Schwichtenberg. An inverse of the evaluation functional for typed lambda-calculus. 1991.

[2] Thorsten Altenkirch, Martin Hofmann, and Thomas Streicher. Categorical reconstruction of a reduction free normalization proof. In *International Conference on Category Theory and Computer Science*, pages 182–199. Springer, 1995.